Name: key

Assessment 9

Math& 264: Multivariable Calculus

Instructions: Please carefully complete these questions by hand. Be sure to show all work including sketching relevant graphs and providing exact answers.

Should you choose to work these on scratch paper, please do not put more than one question on a page. Additional sheets of paper are acceptable. WRITE YOUR NAME ON EVERY PAGE

Upload your solutions to Gradescope by 9 am on Monday (12/7). During your presentation time, you will be asked to explain your thought process and reasoning on one randomly assigned question. Late solutions are available thru 2 pm with a 5% penalty (smaller penalties if you can document that this is a revision and only minor changes).

Please make sure to sign up for your presentation slot. If you are unavailable for any of the times available, please send me a note in Slack and we will find a time that works for you.

https://docs.google.com/spreadsheets/d/17eCs-TpdpqMTu97_csL7NoFjqaO1Qqaj0lxjtMOqYM4/edit?usp=sharing

Reminders: It is okay to collaborate with peers and use online resources. However, the final work should be your own and you should be prepared to present on each question.

(1.1) Is there a field $\vec{G}$ on $\mathbb{R}^3$ such that its curl is $\langle x\sin y, \cos y, z-9xy\rangle$? Explain.

suppose there is $\vec{G}$ s.t. $\nabla\times\vec{F} = \langle x\sin y, \cos y, z-9xy\rangle$

$$\Rightarrow \nabla\cdot(\nabla\times\vec{G}) = \nabla\cdot\langle x\sin y, \cos y, z-9xy\rangle$$

$$= \sin y - \sin y + 1$$

$$= 1$$

$$= 0$$

$\Rightarrow\Leftarrow$

no such $\vec{G}$ exists.

$$\frac{\partial g}{\partial x} = \frac{\partial f}{\partial y} = 3\sqrt{y} - 3\sqrt{y} = 0$$

so conservative

(1.2) Find the work done by the force field $\vec{F} = \left\langle 2y^{3/2}, 3x\sqrt{y} \right\rangle$ in moving an object from $(1,1)$ to $(2,9)$.

① $\int 2y^{3/2}\,dx = 2xy^{3/2} + c(y) = \varphi$

② $\varphi_y = 3x\sqrt{y} + c'(y) = 3x\sqrt{y} \Rightarrow c'(y) = 0$

so $\varphi(x,y) = 2xy^{3/2}$

work $= \varphi(2,9) - \varphi(1,1)$

$= 4(27) - 2(1)$

$= 106$

(1.3) Find the curl and divergence of the field $\vec{F} = \left\langle 6e^{x}\sin y, 5e^{y}\sin z, 8e^{z}\sin x \right\rangle$

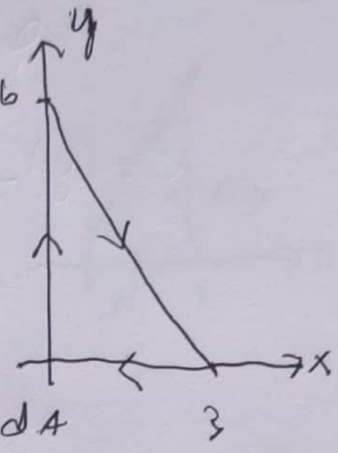
$$\text{curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 6e^{x}\sin y & 5e^{y}\sin z & 8e^{z}\sin x \end{vmatrix}$$

$$= \langle 0 - 5e^{y}\cos z,\ 0 - 8e^{z}\cos x,\ 0 - 6e^{x}\cos y \rangle$$

$$= -\langle 5e^{y}\cos z,\ 8e^{z}\cos x,\ 6e^{x}\cos y \rangle$$

$$\text{div}\,\vec{F} = 6e^{x}\sin y + 5e^{y}\sin z + 8e^{z}\sin x$$

(1.4) Find the circulation of $\vec{F} = \langle y\cos x - xy\sin x, xy + x\cos x \rangle$ about the triangular path from (0,0) to (0,6) to (3,0) and then returning to the origin.



$$\text{circ} = \oint_C \vec{F}\cdot d\vec{r}$$

$$= -\iint_\Delta \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA$$

$$= \iint_\Delta (\cancel{\cos x} - \cancel{x\sin x} - y - \cancel{\cos x} + \cancel{x\sin x})\, dA$$

$$= \int_0^3 \int_0^{6-2x} -y \, dy\, dx$$

$$= -\frac{1}{2}\int_0^3 (6-2x)^2\, dx$$

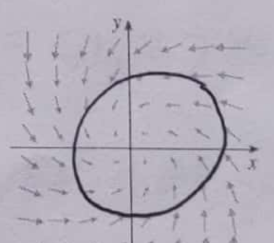
$36 - 24x + 4x^2$

$$= \int_0^3 -18 + 12x - 2x^2\, dx$$

$$= -18(3) + \frac{\overset{6}{\cancel{12}}(9)}{\cancel{2}} - 2\frac{(\overset{9}{\cancel{27}})}{\cancel{3}}$$

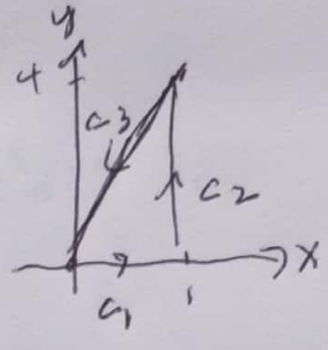
$$= -18$$

(1.5) Is the vector field shown conservative? Please explain.



If the field was conservative, the work around any closed path would be zero. But the tangential components all appear to be CCW (i.e., they won't cancel out). So, no, the field is not conservative

(1.6) Use two different methods to evaluate the line integral $\oint_C xy\,dx + x^2y^3\,dy$ where C is counterclockwise around the triangle with vertices (0,0), (1,0), and (1,4).



Green's Thm

$$\int_0^1\int_0^{4x} 2xy^3 - x \,dy\,dx$$

$$x(2y^3-1)$$

$$= \int_0^1 \left[x\left[\frac{2y^4}{2} - y\right]\right]_0^{4x} dx$$

$$= \int_0^1 \underbrace{x(128x^4 - 4x)}_{128x^5 - 4x^2}\,dx = \frac{128}{6} - \frac{4}{3} = \frac{60}{3} = 20$$

Direct

① parameterize

$\vec{r}_1 = \langle t, 0\rangle$ ON $0 \le t \le 1$ $\quad$ $\vec{r}_2 = \langle 1, t\rangle$ ON $0 \le t \le 4$ $\quad$ $\vec{r}_3 = \langle t, 4t\rangle$ from $t=1$ to $t=0$

② $d\vec{r}_1 = \langle 1, 0\rangle dt \quad d\vec{r}_2 = \langle 0, 1\rangle dt \quad d\vec{r}_3 = \langle 1, 4\rangle dt$

③ work $= \int_0^1 0 + \int_0^4 t^3\,dt + \int_1^0 4t^2\,dt + 64t^5 \cdot 4\,dt$

$$= -\left[\frac{4}{3}t^3 + \frac{256}{6}t^6\right]_0^1 + \left[\frac{t^4}{4}\right]_0^4$$

$$= -\left(\frac{4}{3} + \frac{128}{3}\right) + 64$$

$$= -\frac{132}{3} + 64 = 20$$

$$\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} = -2xe^{-y} + 2xe^{-y} = 0$$

(1.7) Evaluate the line integral $\int_C 2xe^{-y}dx + (2y - x^2e^{-y})dy$ where C is <u>any</u> path from (1,0) to (5,1)

① $\int 2xe^{-y}\,dx = x^2e^{-y} + c(y) = \varphi$

② $\varphi_y = -x^2e^{-y} + c'(y) = 2y - x^2e^{-y}$

$\Rightarrow c'(y) = 2y$

③ $\int 2y\,dy = y^2 + k$

And $\varphi(x,y) = y^2 + x^2e^{-y}$

$$\int_C 2xe^{-y}dx + (2y - x^2e^{-y})dy = \varphi(5,1) - \varphi(1,0)$$
$$= \left(1 + \frac{25}{e}\right) + (0 - 1)$$
$$= \frac{25}{e}$$

(1.8) Determine wither the field $\vec{F} = \langle \overset{f}{e^{yz}}, \overset{g}{xze^{yz}}, \overset{h}{xye^{yz}} \rangle$ is conservative. If it is conservative, find a potential function.

$$\text{curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^{yz} & xze^{yz} & xye^{yz} \end{vmatrix}$$

$$= \langle xe^{yz} + xyze^{yz} - e^{yz}(x + xyz),\ 0,\ 0\rangle$$

$= \vec{0}$ $\therefore \vec{F}$ is conservative.

① $f = e^{yz} \Rightarrow \int e^{yz}\,dx = xe^{yz} + C(y,z) = \varphi$

② $\varphi_y = xze^{yz} + C_y(y,z) = g \Rightarrow C_y(y,z) = 0$ (NO y's IN C)

③ $\int (xze^{yz} + 0)\,dy = xe^{yz} + D(z) = \varphi$

④ $\varphi_z = xye^{yz} + D'(z) = h \Rightarrow D'(z) = 0$ (NO z's IN D)

SO $\varphi(x,y,z) = xe^{yz}$

(1.9) If $\vec{r} = \langle x, y, z \rangle$ and $r = |\vec{r}|$, find ∇r, $\nabla \times \vec{r}$, $\nabla\left(\frac{1}{r}\right)$, and $\nabla \ln r$

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\nabla r = \frac{1}{\sqrt{x^2+y^2+z^2}} \langle x, y, z \rangle$$

$$\nabla \times \vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \langle 0, 0, 0 \rangle$$

$$\nabla \left(\frac{1}{\sqrt{x^2+y^2+z^2}} \right) = -\frac{1}{(x^2+y^2+z^2)^{3/2}} \langle x, y, z \rangle$$

$$\nabla \ln \sqrt{x^2+y^2+z^2} = \nabla \frac{1}{2} \ln (x^2+y^2+z^2)$$

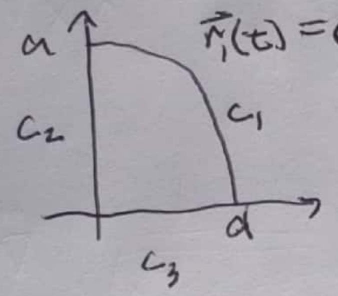
$$= \frac{\langle x, y, z \rangle}{x^2+y^2+z^2}$$

(1.10) Use Green's Theorem to find the centroid of the quarter-circular region of radius *a*.

$$\text{Area} = \frac{1}{4}\pi a^2$$

$$M_y = \iint_D x \, dA = \frac{1}{2}\oint_C x^2 \, dy$$

parameterize



$\vec{r}_1(t) = \langle a\cos(t), a\sin(t)\rangle$ on $0 \le t \le \frac{\pi}{2}$

$d\vec{r}_1 = a\langle -\sin t, \cos t\rangle \, dt$

$$M_y = \frac{1}{2}\int_0^{\frac{\pi}{2}} a^2\cos^2 t \cdot a\cos t \, dt$$

$$+$$

$$\frac{1}{2}\int_0^a 0 \;\underline{\hspace{2em}}$$

$x = 0$

$$+$$

$$\frac{1}{2}\int_0^a \underline{\hspace{1em}} \cdot \underset{\sim}{0} \, dt$$

dy

$$= \frac{a^3}{2}\int_0^{\pi/2} (1 - \sin^2 t)\cos t \, dt$$

$$= \frac{a^3}{2}\left(1 - \left[\frac{\sin^3 t}{3}\right]_0^{\pi/2}\right)$$

$$= \frac{a^3}{3} \quad \text{AND} \quad \bar{x} = \frac{\frac{a^3}{3}}{\frac{\pi a^2}{4}} = \frac{4a}{3\pi}$$

$$\text{centroid} = \left(\frac{4a}{3\pi}, \frac{4a}{3\pi}\right)$$