Name: Key

Assessment 8

Math& 264: Multivariable Calculus

Instructions: Please carefully complete these questions by hand. Be sure to show all work including sketching relevant graphs and providing exact answers.

Should you choose to work these on scratch paper, please do not put more than one question on a page. Additional sheets of paper are acceptable. WRITE YOUR NAME ON EVERY PAGE

Upload your solutions to Gradescope by 9 am on Monday (11/30). During your presentation time, you will be asked to explain your thought process and reasoning on one randomly assigned question. Late solutions are available thru 2 pm with a 5% penalty (smaller penalties if you can document that this is a revision and only minor changes).

Please make sure to sign up for your presentation slot. If you are unavailable for any of the times available, please send me a note in Slack and we will find a time that works for you.

https://docs.google.com/spreadsheets/d/17eCs-TpdpqMTu97_csL7NoFjqaO1Qqaj0lxjtMOqYM4/edit?usp=sharing

Reminders: Show all steps (**including in evaluating limits**). Thanksgiving Bonus: You only need to solve nine questions. For the bonus question, state one thing you are thankful for and receive full credit☺.

(1.1) Test the series $\sum_{n=1}^{\infty} \frac{1}{n+n\cos^2(8n)}$ for convergence or divergence. If it converges, does the series converge conditionally or absolutely? Be sure to justify your work.[1]

$$\underbrace{\sum_{n=1}^{\infty} \frac{1}{n+1n}} \leq \sum_{n=1}^{\infty} \frac{1}{n+n\cos^2 8n}$$

$\frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n}$ which is the divergent harmonic series.

$\therefore$ the series $\sum_{n=1}^{\infty} \frac{1}{n+n\cos^2(8n)}$ diverges by the comparison test.

[1] For tests requiring it, you do not need to verify that sequences are decreasing. This holds true throughout the Assessment.

(1.2) Test the series $\sum_{j=1}^{\infty} \frac{(-1)^j \sqrt{j}}{j+9}$ for convergence or divergence. If it converges, does the series converge conditionally or absolutely? Be sure to justify your work.

$\lim_{j\to\infty} \frac{\sqrt{j}}{j+9} = 0$ so $\sum_{j=1}^{\infty} \frac{(-1)^j \sqrt{j}}{j+9}$ converges by the A.S.T.

$\lim_{j\to\infty} \frac{\frac{1}{\sqrt{j}}}{\frac{\sqrt{j}}{j+9}} = 1.$ since $\sum_{j=1}^{\infty} \frac{1}{\sqrt{j}}$ is a divergent p series, we know the series does not converge absolutely by the L.C.T.

$\therefore$ The series $\sum_{j=1}^{\infty} \frac{(-1)^j \sqrt{j}}{j+9}$ converges conditionally by the A.S.T.

(1.3) Find the gradient vector field of $f(x,y,z) = 6\sqrt{x^2+y^2+z^2}$

$$\nabla f = \frac{3}{\sqrt{x^2+y^2+z^2}} \langle 2x, 2y, 2z \rangle$$

(1.4) Find the work done by the force field $\langle x-y^2, y-z^2, z-x^2\rangle$ on a particle that moves along the line segment from (0,0,1) to (3,1,0).

Step 1: parameterize

$$\vec{r}(t) = t\langle 3,1,0\rangle + (1-t)\langle 0,0,1\rangle$$

ON $0 \leq t \leq 1$

$$= \langle 3t, t, 1-t\rangle$$

Step 2: $d\vec{r} = \langle 3, 1, -1\rangle dt$

Step 3: sub and evaluate.

$$\text{work} = \int_C \vec{F}\cdot d\vec{r}$$

$$= \int_0^1 \langle 3t-t^2, \underbrace{t-(1-t)^2}_{-1+3t-t^2}, 1-t-9t^2\rangle \cdot \langle 3,1,-1\rangle dt$$

$$= \int_0^1 9t-3t^2-1+3t-t^2-1+t+9t^2 \, dt$$

$$= \int_0^1 -2+13t+5t^2 \, dt$$

$$= -2+\frac{13}{2}+\frac{5}{3}$$

$$= \frac{37}{6}$$

(1.5) Determine whether the series $\sum_{k=1}^{\infty} \frac{k\ln k}{(k+2)^3}$ is convergent or divergent. If it converges, does the series converge conditionally or absolutely? Be sure to justify your work.

$$\frac{k\ln k}{(k+2)^3} \le \frac{k\ln k}{k^3} = \frac{\ln k}{k^2}$$

AND $\int_1^\infty \frac{\ln x}{x^2}\,dx = \lim_{t\to\infty}\int_1^t \frac{\ln x}{x^2}\,dt$

Let $u = \ln x$

$du = \frac{1}{x}dx$

$dv = \frac{1}{x^2}dx$

$v = -\frac{1}{x}$

$$= \lim_{t\to\infty}\left[-\frac{\ln x}{x} + \int \frac{1}{x^2}\,dx\right]_1^t$$

$$= \lim_{t\to\infty}\left[-\frac{\ln x}{x} - \frac{1}{x}\right]_1^t$$

$$= \lim_{t\to\infty}\left(1 - \frac{\ln t + 1}{t}\right)$$

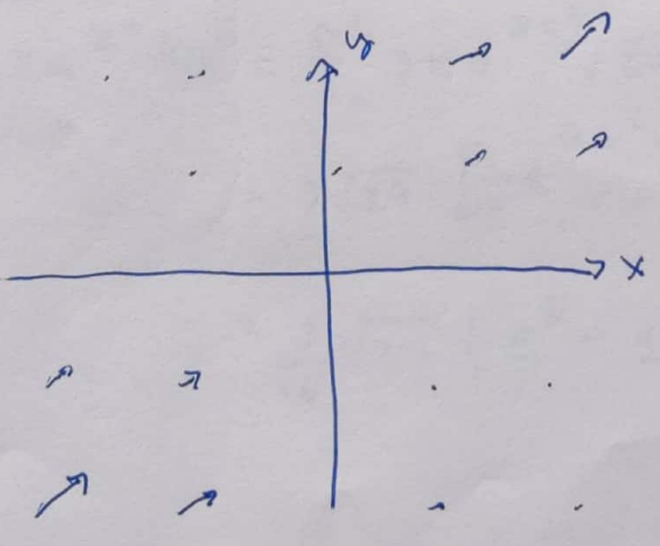
$$\overset{H}{=} 1 - \lim_{t\to\infty}\left(-\frac{1}{t}\right) = 1$$

$\therefore$ Since $\sum_{k=1}^{\infty}\frac{\ln k}{k^2}$ converges by the integral test, we know $\sum_{k=1}^{\infty}\frac{k\ln k}{(k+2)^3}$ converges by the comparison test.

(1.6) Find and sketch the gradient vector field of $g(x,y) = xe^{3xy}$. You may use technology to help with graphing.

$$\nabla g(x,y) = \langle e^{3xy} + x\cdot 3y e^{3y},\ 3x^2 e^{3xy}\rangle$$

$$= e^{3xy}\langle 1 + 3xy,\ 3x^2\rangle$$



Hard to see. Must be close to the origin.

(1.7) Test the series $\sum_{n=1}^{\infty} \tan\left(\frac{3}{n}\right)$ for convergence or divergence. If it converges, does the series converge conditionally or absolutely? Be sure to justify your work.

$$\lim_{N\to\infty} \frac{\tan\left(\frac{3}{N}\right)}{\frac{1}{N}} = \lim_{x\to\infty} \frac{\tan\frac{3}{x}}{\frac{1}{x}}$$

$$\overset{H}{=} \lim_{x\to\infty} \frac{\sec^2\left(\frac{3}{x}\right)\cdot -\frac{3}{x^2}}{-\frac{1}{x^2}}$$

$$= 3\sec^2(0) = 3.$$

$\therefore$ since $\sum_{N=1}^{\infty} \frac{1}{N}$ diverges, $\sum_{N=1}^{\infty} \tan\frac{3}{N}$ also diverges by the limit comparison test.

(1.8) Evaluate the line integral $\int_C xe^{yz}\,ds$ where C is the line segment from (0,0,0) to (3,4,2).

Step 1: parameterize
$\vec{r}(t) = \langle 3t, 4t, 2t\rangle$ on $0 \le t \le 1$

Step 2: $ds = \sqrt{9+16+4}\,dt = \sqrt{29}\,dt$

Step 3: sub and evaluate.

$$\int_C xe^{yz}\,ds = \int_0^1 3te^{8t^2}\sqrt{29}\,dt$$

$$= 3\sqrt{29}\left[\frac{1}{16}e^{8t^2}\right]_0^1$$

$$= \frac{3}{16}\sqrt{29}\left(e^8 - 1\right)$$

(1.9) A thin wire is bent into the shape of a semicircle $x^2+y^2=100, x\geq 0$. If the linear density is a constant k, find the mass and center of mass of the wire.

$$\text{Mass} = \int_C k\, ds$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 10k\, dt$$

$$= 10k\pi$$

Step 1: parameterize

$$\vec{r}(t) = \langle 10\cos t, 10\sin t\rangle$$

on $-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$

Step 2: ds

$$ds = \sqrt{(-10\sin t)^2 + (10\cos t)^2}\, dt$$

$$= 10\, dt$$

$M_x = 0$ (by symmetry)

$$M_y = \int_C k\, x\, ds$$

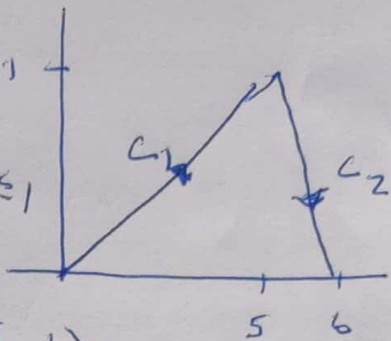
$$= \int_{-\pi/2}^{\pi/2} k \cdot 10\cos(t)(10)\, dt$$

$$= 200k$$

$$(\bar{x}, \bar{y}) = \left(\frac{200k}{10k\pi}, 0\right) = \left(\frac{20}{\pi}, 0\right)$$

(1.10) Evaluate the line integral $\int_C (x+5y)\,dx + x^2 dy$ where C consists of line segments from (0,0) to (5,1) and (5,1) to (6,0).

<u>Step 1:</u> parameterize.



$C_1: \vec{r}_1(t) = \langle 5t, t \rangle$ on $0 \le t \le 1$

$C_2: \vec{r}_2(s) = s\langle 6, 0\rangle + (1-s)\langle 5, 1\rangle$

$= \langle s+5, 1-s \rangle$ on $0 \le s \le 1$

<u>Step 2:</u> $d\vec{r}_1 = \langle 5, 1\rangle\, dt$

$d\vec{r}_2 = \langle 1, -1 \rangle\, ds$

<u>Step 3:</u> sub & evaluate

$$\int_C (x+5y)\,dx + x^2 dy$$

$$= \int_{C_1} (x+5y)\,dx + x^2 dy + \int_{C_2} (x+5y)\,dx + x^2 dy$$

$$= \int_0^1 [(5t+5t)(5) + 25t^2(1)]\,dt + \int_0^1 [(s+5+\underbrace{5(1-s)}_{5-5s})(1) + \underbrace{(s+5)^2}_{-s^2-10s-25}(-1)]\,ds$$

$$= \int_0^1 (50t + 25t^2)\,dt + \int_0^1 -15 - 14s - s^2\, ds$$

$$= 25 + \frac{25}{3} - 15 - 7 - \frac{1}{3}$$

$$= 11$$