Name: Key

**Assessment 6**

**Math& 264: Multivariable Calculus**

Instructions: Please carefully complete these questions by hand. Be sure to show all work including sketching relevant graphs and providing exact answers.

Should you choose to work these on scratch paper, please do not put more than one question on a page. Additional sheets of paper are acceptable. WRITE YOUR NAME ON EVERY PAGE

Upload your solutions to Gradescope by 9 am on Monday (11/9). During your presentation time, you will be asked to explain your thought process and reasoning on one randomly assigned question. Late solutions are available thru 2 pm with a 5% penalty (smaller penalties if you can document that this is a revision and only minor changes).

Please make sure to sign up for your presentation slot. If you are unavailable for any of the times available, please send me a note in Slack and we will find a time that works for you.

https://docs.google.com/spreadsheets/d/17eCs-TpdpqMTu97_csL7NoFjqaO1Qqaj0IxjtMOqYM4/edit?usp=sharing

Pro tip: Make sure to check out the honors videos ... and consider creating your own.

(1.1) Determine whether the series $\sum_{n=1}^{\infty} \frac{\ln(n)}{n^7}$ is convergent or divergent. Use any method developed in this course. Be sure to justify your work.[1]

scratch work $\int \frac{\ln x}{x^7}\,dx = -\frac{1}{6}\frac{\ln x}{x^6} + \int \frac{1}{6}\cdot\frac{1}{x^6}\,\frac{1}{x}\,dx = -\frac{\ln x}{6x^6} - \frac{1}{36x^6}$

Let $u = \ln x$ $\quad dv = \frac{dx}{x^7}$

$du = \frac{dx}{x}$ $\quad v = -\frac{1}{6}x^{-6}$

$\int_1^\infty \frac{\ln x}{x^7} = \lim_{t\to\infty} -\frac{1}{6}\left[\frac{\ln x}{x^6} + \frac{1}{6x^6}\right]_1^t$

$= -\frac{1}{6}\lim_{t\to\infty}\left(\frac{\ln t}{t^6} + \cancel{\frac{1}{6t^6}}^{\,0} - \frac{1}{6}\right)$

$\overset{H}{=} -\frac{1}{6}\left[\left(\lim_{t\to\infty}\cancel{\frac{1}{t\cdot 6t^5}}^{\,0}\right) - \frac{1}{6}\right]$

$= \frac{1}{36}$

Since $\int_1^\infty \frac{\ln x}{x^7}\,dx$ converges, we know $\sum_{n=1}^{\infty}\frac{\ln(n)}{n^7}$ also converges by the integral test.

[1] For tests requiring it, you do not need to verify that sequences are decreasing. This holds true throughout the Assessment.

(1.2) Determine whether the series $\sum_{n=1}^{\infty} \frac{2}{n^2+81}$ is convergent or divergent. Use any method developed in this course. Be sure to justify your work.

$$\lim_{n \to \infty} \frac{\frac{1}{n^2}}{\frac{2}{n^2+81}} = \lim_{n \to \infty} \frac{n^2+81}{2n^2} = \frac{1}{2} \text{ (non-zero constant)}$$

since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (p-series) we know

$\sum_{n=1}^{\infty} \frac{2}{n^2+81}$ converges by the L.C.T.

(1.3) Determine whether the series $\underbrace{\frac{1}{4}+\frac{1}{7}+\frac{1}{10}+\frac{1}{13}}_{\sum_{n=1}^{\infty} \frac{1}{3n+1}}+\ldots$ is convergent or divergent. Use any method developed in this course. Be sure to justify your work.

$$\lim_{n \to \infty} \frac{\frac{1}{n}}{\frac{1}{3n+1}} = \lim_{n \to \infty} \frac{3n+1}{n} = 3 \text{ (non-zero constant)}$$

since the harmonic series diverges,

$\sum_{n=1}^{\infty} \frac{1}{3n+1}$ also diverges by the L.C.T.

(1.4) The Euler-Riemann zeta function is $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$. Euler showed that $\zeta(4) = \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$. Use this result to find (a.) $\sum_{n=1}^{\infty} \frac{3}{n^4}$ and (b.) $\sum_{n=7}^{\infty} \frac{3}{(n-5)^4}$. Clearly show your work.

(a) $\sum_{n=1}^{\infty} \frac{3}{n^4} = 3\sum_{n=1}^{\infty} \frac{1}{n^4} = 3\frac{\pi^4}{90} = \frac{\pi^4}{30}$

(b) $\sum_{n=7}^{\infty} \frac{3}{(n-5)^4} = 3\left(\frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \dots\right)$

Add a special zero

$= 3\underbrace{\left(\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \dots\right)}_{\frac{\pi^4}{90}} - 3\cdot\frac{1}{1^4}$

$= \frac{\pi^4}{30} - 3$

(1.5) Find the values of $p$ for which the series $\sum_{n=2}^{\infty} \frac{9}{n(\ln(n))^p}$ is convergent.

$$\int_2^{\infty} \frac{9}{x(\ln x)^p}\,dx = \lim_{t\to\infty} \int_2^{t} \frac{9}{x(\ln x)^p}\,dx$$

$9u^{-p}$

Let $u = \ln x$

$du = \frac{dx}{x}$

$$= \lim_{t\to\infty} \int_{\ln 2}^{\ln t} \frac{9}{u^p}\,du$$

$$= \lim_{t\to\infty} \begin{cases} \left[\frac{9}{1-p} u^{1-p}\right]_{\ln 2}^{\ln t}, & p<1 \\ \left[9\ln|u|\right]_{\ln 2}^{\ln t}, & p=1 \\ \left[\frac{9}{1-p} u^{1-p}\right]_{\ln 2}^{\ln t}, & p>1 \end{cases}$$

$$= \begin{cases} \text{diverges}, & p \leq 1 \\ \text{converges}, & p>1 \end{cases}$$

(1.6) Determine whether the series $\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$ is convergent or divergent. Use any method developed in this course. Be sure to justify your work.

$$\sum_{n=1}^{\infty} \frac{\ln n}{n} \geq \sum_{n=1}^{\infty} \frac{1}{n}$$ which is the divergent hamonic series.

$\therefore \sum_{n=1}^{\infty} \frac{\ln n}{n}$ diverges by the comparison test.

(1.7) Determine whether the series $\sum_{n=1}^{\infty} \frac{e^{6/n}}{n}$ is convergent or divergent. Use any method developed in this course. Be sure to justify your work.

$$e^{6/n} > e^{0} \quad \text{on } n > 0$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{e^{6/n}}{n} \geq \sum_{n=1}^{\infty} \frac{1}{n}$$ which is the divergent harmonic series.

$\therefore \sum_{n=1}^{\infty} \frac{e^{6/n}}{n}$ diverges by the comparison test.

(1.8) Determine whether the series $\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{3} e^{-3n}$ is convergent or divergent. Use any method developed in this course. Be sure to justify your work.

$$\lim_{n\to\infty} \frac{(1+\frac{1}{n})^3 e^{-3n}}{e^{-3n}} = \lim_{n\to\infty} (1+\frac{1}{n})^3 = 1$$ (non-zero constant)

And $\sum_{n=1}^{\infty} e^{-3n} = \sum_{n=1}^{\infty} (\frac{1}{e^3})^n$ is a convergent geometric series

$\therefore \sum_{n=1}^{\infty} (1+\frac{1}{n})^3 e^{-3n}$ converges by the L.C.T.

(1.9) Determine whether the series $\sum_{n=1}^{\infty} \frac{n^2 - 4n}{n^3 + 6n + 4}$ is convergent or divergent. Use any method developed in this course. Be sure to justify your work.

$$\lim_{n \to \infty} \frac{\frac{1}{n}}{\frac{n^2 - 4n}{n^3 + 6n + 4}} = \lim_{n \to \infty} \frac{n^3 + 6n + 4}{n^3 - 4n^2} = 1 \quad \left(\begin{array}{c}\text{non-zero}\\ \text{constant}\end{array}\right)$$

since $\sum_{n=1}^{\infty} \frac{1}{n}$ is the divergent harmonic series,

we conclude $\sum_{n=1}^{\infty} \frac{n^2 - 4n}{n^3 + 6n + 4}$ diverges by

the limit comparison test.

(1.10) Determine whether the series $\sum_{n=1}^{\infty} \frac{\sqrt{n+3}}{4n^2 + n + 4}$ is convergent or divergent. Use any method developed in this course. Be sure to justify your work.

$$\lim_{n \to \infty} \frac{\frac{1}{n^{3/2}}}{\frac{\sqrt{n+3}}{4n^2 + n + 4}} = \lim_{n \to \infty} \frac{4n^2 + n + 4}{n^{3/2} \sqrt{n+3}} = 4 \quad \left(\begin{array}{c}\text{non-zero}\\ \text{constant}\end{array}\right)$$

since $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ is a convergent p-series,

$\sum_{n=1}^{\infty} \frac{\sqrt{n+3}}{4n^2 + n + 4}$ converges by the limit

comparison test.