Name: key

Assessment 4

Math& 264: Multivariable Calculus

Instructions: Please carefully complete these questions by hand. Be sure to show all work including sketching relevant graphs and providing exact answers.

Should you choose to work these on scratch paper, please do not put more than one question on a page. Additional sheets of paper are acceptable. WRITE YOUR NAME ON EVERY PAGE

Upload your solutions to Gradescope by 9 am on Monday (10/26). During your presentation time, you will be asked to explain your thought process and reasoning on one randomly assigned question. Late solutions are available thru 2 pm with a 5% penalty (smaller penalties if you can document that this is a revision and only minor changes).

Please make sure to sign up for your presentation slot. If you are unavailable for any of the times available, please send me a note in Slack and we will find a time that works for you.

https://docs.google.com/spreadsheets/d/17eCs-TpdpqMTu97_csL7NoFjqaO1Qqaj0IxjtMOqYM4/edit?usp=sharing

Pro tip: Make sure to check out the honors videos ... and consider creating your own.

(1.1) Evaluate $I = \iiint_E (x^2+y^2)\,dV$ where E lives between the spheres $x^2+y^2+z^2=9$ and $x^2+y^2+z^2=16$.

$$I = \int_0^{2\pi}\int_0^{\pi}\int_3^4 \left[(\rho\sin\phi\cos\theta)^2 + (\rho\sin\phi\cos\theta)^2\right]\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$= 2\pi\int_0^{\pi}\int_3^4 \rho^4\sin^3\phi\,d\rho\,d\phi$$

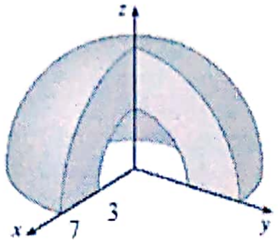
$$= 2\pi\left[\frac{\rho^5}{5}\right]_3^4 \int_0^{\pi}\sin\phi\,(1-\cos^2\phi)\,d\phi$$

$$= \frac{2\pi}{5}(4^5-3^5)\underbrace{\left[-\cos\phi + \frac{\cos^3\phi}{3}\right]_0^{\pi}}_{1-\frac{1}{3}+1-\frac{1}{3}}$$

$$= \frac{8\pi}{15}$$

$$= \frac{6248\pi}{15} \checkmark$$

(1.2) Set up five multiple integrals to describe the volume of the region between the two spherical shells as pictured.



Rectangular coordinates (double integral)

$$V = 3\left[\int_{-7}^{0}\int_{0}^{\sqrt{49-x^2}} \sqrt{49-x^2-y^2}\, dy\, dx - \int_{-3}^{0}\int_{0}^{\sqrt{9-x^2}} \sqrt{9-x^2-y^2}\, dy\, dx\right]$$

Rectangular coordinates (triple integral)

$$V = 3\left[\int_{-7}^{0}\int_{0}^{\sqrt{49-x^2}}\int_{0}^{\sqrt{49-x^2-y^2}} 1\, dz\, dy\, dx - \int_{-3}^{0}\int_{0}^{\sqrt{9-x^2}}\int_{0}^{\sqrt{9-x^2-y^2}} 1\, dz\, dy\, dx\right]$$

Polar coordinates

$$V = \int_{\frac{\pi}{2}}^{2\pi}\int_{0}^{7} \sqrt{49-r^2}\cdot r\, dr\, d\theta - \int_{\frac{\pi}{2}}^{2\pi}\int_{0}^{3} \sqrt{9-r^2}\, r\, dr\, d\theta$$

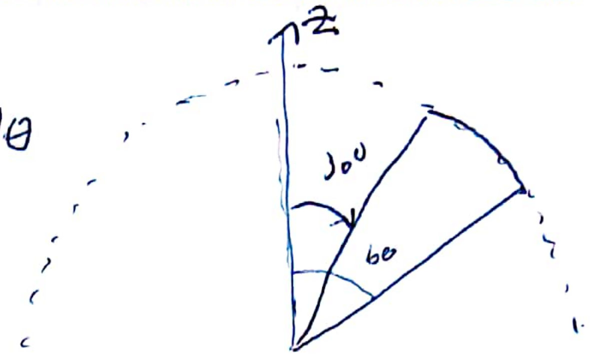
Cylindrical coordinates

$$V = \int_{\frac{\pi}{2}}^{2\pi}\int_{0}^{7}\int_{0}^{\sqrt{49-r^2}} 1\, r\, dz\, dr\, d\theta - \int_{\frac{\pi}{2}}^{2\pi}\int_{0}^{3}\int_{0}^{\sqrt{9-r^2}} 1\, r\, dz\, dr\, d\theta$$

Spherical coordinates

$$V = \int_{\frac{\pi}{2}}^{2\pi}\int_{0}^{\frac{\pi}{2}}\int_{3}^{7} 1\, \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

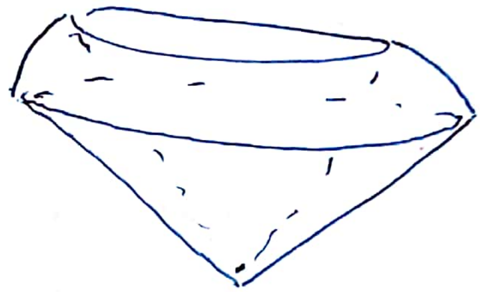
(1.3) Find the volume of the part of the ball with radius 6 that lies between the inclinations of 30 and 60 degrees.



$$V = \int_0^{2\pi}\int_{\pi/6}^{\pi/3}\int_0^6 1\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= 2\pi \left[\frac{\rho^3}{3}\right]_0^6 \int_{\pi/6}^{\pi/3} \sin\phi \, d\phi$$

$$= 2\pi \cdot \frac{216}{3} \underbrace{\left[-\cos\phi\right]_{\pi/6}^{\pi/3}}_{-\frac{1}{2} + \frac{\sqrt{3}}{2}}$$

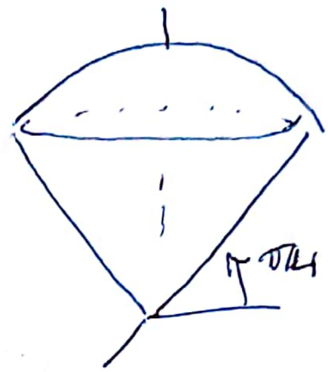


$$= \frac{216\pi}{3}(\sqrt{3}-1)$$

$$= 72\pi(\sqrt{3}-1) \checkmark$$

(1.4) Find the centroid of the solid with uniform density that lies above the cone $z=\sqrt{x^2+y^2}$ and below the sphere $x^2+y^2+z^2=9$.

$$m = \int_0^{2\pi}\int_0^{\pi/4}\int_0^{3} k\,\rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$



$$= 2\pi k \underbrace{\left[\frac{\rho^3}{3}\right]_0^3}_{9} \underbrace{\left[-\cos\phi\right]_0^{\pi/4}}_{1-\frac{\sqrt{2}}{2}}$$

$$= 9\pi(2-\sqrt{2})k$$

$$m_{xy} = \int_0^{2\pi}\int_0^{\pi/4}\int_0^{3} k\,\overbrace{\rho\cos\phi}^{z}\,\rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$

$$= 2\pi k \left[\frac{\rho^4}{4}\right]_0^3 \left[\frac{\sin^2\phi}{2}\right]_0^{\pi/4}$$

$$= \frac{81}{4}\pi k \cdot \left(\frac{1}{2}-0\right) = \frac{81\pi k}{8}$$

$$m_{yz} = \int_0^{2\pi}\int_0^{\pi/4}\int_0^{3} k\,\overbrace{\rho\sin\phi\cos\theta}^{x}\,\rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\cos\theta\, d\theta \cdot \text{stuff} = 0$$

by symmetry, $m_{xz} = 0$

so the centroid is $\left(0, 0, \frac{9}{8(2-\sqrt{2})}\right)$ ✓

(1.5) Evaluate $I = \int_{-5}^{5} \int_{-\sqrt{25-y^2}}^{\sqrt{25-y^2}} \int_{-\sqrt{25-x^2-y^2}}^{\sqrt{25-x^2-y^2}} \underbrace{(x^2 z + y^2 z + z^3)}_{z(x^2+y^2+z^2)} \, dz \, dx \, dy$

$$I = \int_0^{2\pi} \int_0^{\pi} \int_0^{5} \rho\cos\phi \cdot \rho^2 \, \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

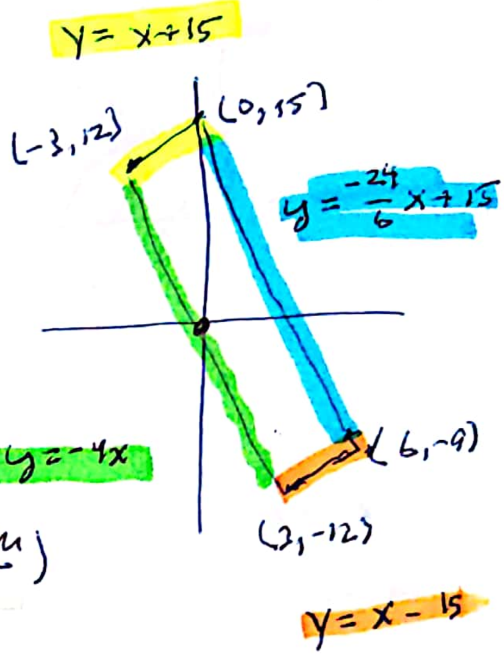
$\boxed{\text{This is a sphere w/ radius 5.}}$

$$= 2\pi \int_0^{\pi} \sin\phi\cos\phi \, d\phi \int_0^{5} \rho^5 \, d\rho$$

$$= \pi \left[\sin^2\phi\right]_0^{\pi} \cdot \text{stuff}$$

$$= 0 \checkmark$$

(1.6) Evaluate $I = \iint_R (20x + 15y)\, dA$ where R is the parallelogram with vertices (-3,12), (3,-12), (6,-9), and (0,15).



Let $u = y - x$

$v = 4x + y$

$x = \dfrac{v-u}{5}$ and $y = \dfrac{v+4u}{5}$

so $20x + 15y = 20\left(\dfrac{v-u}{5}\right) + 15\left(\dfrac{v+4u}{5}\right)$

$= 4v - 4u + 3v + 12u$

$= 7v + 8u$

scaling factor $= \left|\dfrac{\partial(x,y)}{\partial(u,v)}\right| = \dfrac{1}{\left|\dfrac{\partial(u,v)}{\partial(x,y)}\right|} = \dfrac{1}{5}$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} -1 & 1 \\ 4 & 1 \end{vmatrix} = -5$$

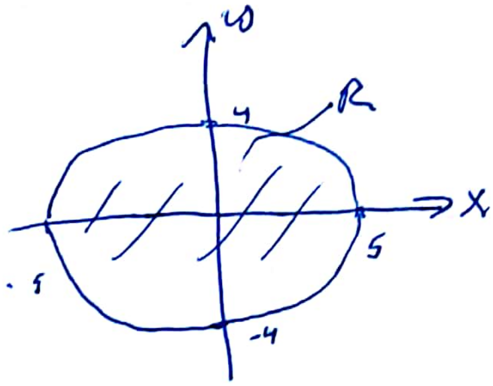
so $I = \displaystyle\int_{-15}^{15}\int_{0}^{15} (7v + 8u)\frac{1}{5}\, dv\, du$

$= \dfrac{1}{5}\displaystyle\int_{-15}^{15} \left[\frac{7}{2}v^2 + 8uv\right]_0^{15} du$

$= \dfrac{1}{5}\displaystyle\int_{-15}^{15} \frac{1575}{2} + 120u\, du$

$= \dfrac{1575(30)}{5(2)} = 4725$

(1.7) Evaluate $I = \iint_R 6x^2\, dA$ where R is the region bounded by the ellipse $16x^2 + 25y^2 = 400$



Let $u = 4x$ and $v = 5y$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 4 & 0 \\ 0 & 5 \end{vmatrix} = 20$$

$$\Rightarrow \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left|\frac{1}{20}\right| = \frac{1}{20}$$

Let $u = r\cos\theta$

$v = r\sin\theta$

$$\Rightarrow I = 6\cdot\frac{1}{16}\iint_{u^2+v^2\le 400} u^2 \cdot \frac{1}{20}\, dA'$$

$$= \frac{3}{160}\int_0^{2\pi}\int_0^{20} (r\cos\theta)^2\, r\, dr\, d\theta$$

$$= \frac{3}{160}\int_0^{2\pi} \frac{1+\cos 2\theta}{2}\, d\theta \int_0^{20} r^3\, dr$$

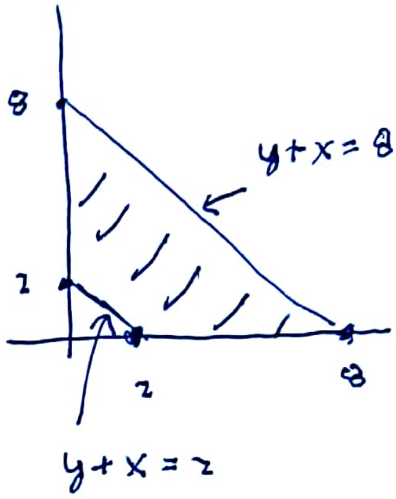
$$= \frac{3}{320}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{2\pi}\left[\frac{r^4}{4}\right]_0^{20}$$

$$= \frac{3\cdot 2\pi}{1280}\cdot 20^4$$

$$= 750\pi \checkmark$$

(1.8) Evaluate $I = \iint_R 7\cos\left(5 \cdot \frac{y-x}{y+x}\right) dA$ where R is the trapezoidal region with vertices (2,0), (8,0), (0,8), and (0,2).

Let $u = y - x$ and $v = y + x$



$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} = -2$$

$$\Rightarrow \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \frac{1}{2}$$

$$I = \int_2^8 \int_{-v}^{v} 7\cos\left(5\frac{u}{v}\right) \cdot \frac{1}{2}\, du\, dv$$

$$= \frac{7}{2}\int_2^8 \left[\frac{v}{5}\sin\left(5\frac{u}{v}\right)\right]_{-v}^{v} dv$$

$$= \frac{7}{10}\int_2^8 v\left(\sin(5) - \sin(-5)\right) dv$$

$$= \frac{14\sin(5)}{10}\int_2^8 v\, dv$$

$$= \frac{7}{5}\sin(5)\left[\frac{v^2}{2}\right]_2^8$$

$$= \frac{7}{10}\sin(5)(60)$$

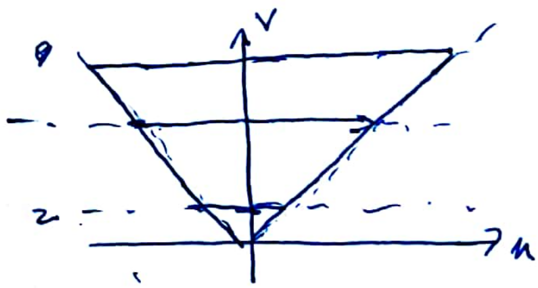
$$= 42\sin(5) \checkmark$$

$x = 0 \Rightarrow \frac{v-u}{2} = x = 0$

$\Rightarrow v = u$

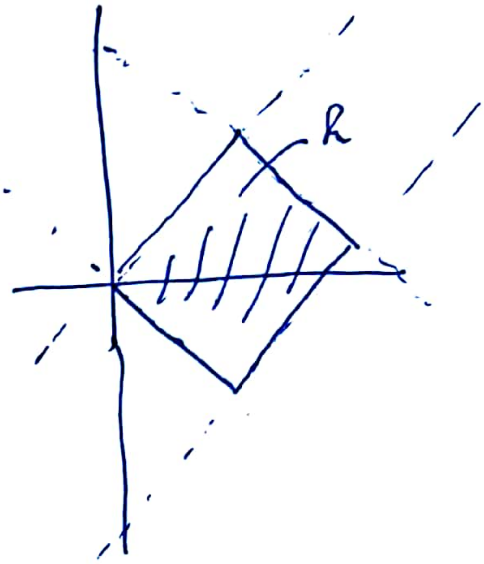
$y = 0 \Rightarrow \frac{v+u}{2} = y = 0$

$\Rightarrow v = -u$



(1.9) Evaluate $I=\iint_R 3(x+y)e^{x^2-y^2}\,dA$ where R is the rectangle enclosed by $x-y=0$, $x-y=4$, $x+y=0$, and $x+y=4$.

Let $u = x-y$ and $v = x+y$



$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 2$$

$$\Rightarrow \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \frac{1}{2}$$

and
$$I = \int_0^4\int_0^4 3v\,e^{uv}\cdot\tfrac{1}{2}\,du\,dv$$

$$= \frac{3}{2}\int_0^4 \left[e^{uv}\right]_0^4 dv$$

$$= \frac{3}{2}\int_0^4 (e^{4v}-1)\,dv$$

$$= \frac{3}{2}\left[\frac{1}{4}e^{4v}-v\right]_0^4$$

$$= \frac{3}{2}\left[\frac{1}{4}(e^{16}-1)-4\right]$$

$$= \frac{3}{8}(e^{16}-17) \checkmark$$

(1.10) Use the Jacobian to derive the scaling factor when going from rectangular to spherical coordinates.

$x = \rho \cos\theta \sin\phi$; $y = \rho \sin\theta \sin\phi$; $z = \rho \cos\phi$.

$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} = \begin{vmatrix} \cos\theta\sin\phi & -\rho\sin\theta\sin\phi & \rho\cos\theta\cos\phi \\ \sin\theta\sin\phi & \rho\cos\theta\sin\phi & \rho\sin\theta\cos\phi \\ \cos\phi & 0 & -\rho\sin\phi \end{vmatrix}$$

$$= \cos\phi \begin{vmatrix} -\rho\sin\theta\sin\phi & \rho\cos\theta\cos\phi \\ \rho\cos\theta\sin\phi & \rho\sin\theta\cos\phi \end{vmatrix}$$

$$- \rho\sin\phi \begin{vmatrix} \cos\theta\sin\phi & -\rho\sin\theta\sin\phi \\ \sin\theta\sin\phi & \rho\cos\theta\sin\phi \end{vmatrix}$$

$$= -\rho^2 \sin\phi\cos^2\phi\,(\sin^2\theta + \cos^2\theta) - \rho^2\sin^3\phi\,(\cos^2\theta + \sin^2\theta)$$

$$= -\rho^2 \sin\phi\,(\cos^2\phi + \sin^2\phi)$$

$$= -\rho^2\sin\phi$$

So the scaling factor $= \left|\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)}\right| = \rho^2\sin\phi$

assuming $0 \le \phi \le \pi$.