Name: Key

Assessment 3

Math& 264: Multivariable Calculus

Instructions: Please carefully complete these questions by hand. Be sure to show all work including sketching relevant graphs and providing exact answers.

Should you choose to work these on scratch paper, please do not put more than one question on a page. Additional sheets of paper are acceptable. WRITE YOUR NAME ON EVERY PAGE

Upload your solutions to Gradescope by 9 am on Monday (10/19). During your presentation time, you will be asked to explain your thought process and reasoning on one randomly assigned question. Late solutions are available thru 2 pm with a 5% penalty (smaller penalties if you can document that this is a revision and only minor changes).

Please make sure to sign up for your presentation slot. If you are unavailable for any of the times available, please send me a note in Slack and we will find a time that works for you.

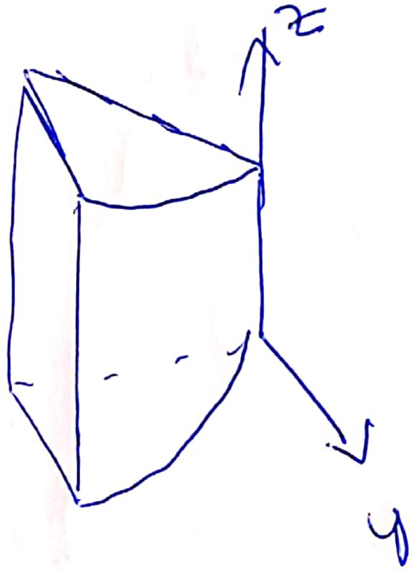
https://docs.google.com/spreadsheets/d/17eCs-TpdpqMTu97_csL7NoFjqaO1Qqaj0lxjtMOqYM4/edit?usp=sharing

Pro tip: Make sure to check out the honors videos ... and consider creating your own.

(1.1) Evaluate the triple integral $I=\iiint_E 3xy\,dV$ where *E* lies under the plane $z=1+x+y$ and above the region in the *xy*-plane bounded by the curves $y=\sqrt{x}$, $y=0$, and $x=1$.

$$I=\int_0^1\int_0^{\sqrt{x}}\underbrace{\int_0^{1+x+y} 3xy\,dz}_{3xy(1+x+y)}\,dy\,dx$$

$$=\int_0^1\int_0^{\sqrt{x}} 3xy+3x^2y+3xy^2\,dy\,dx$$



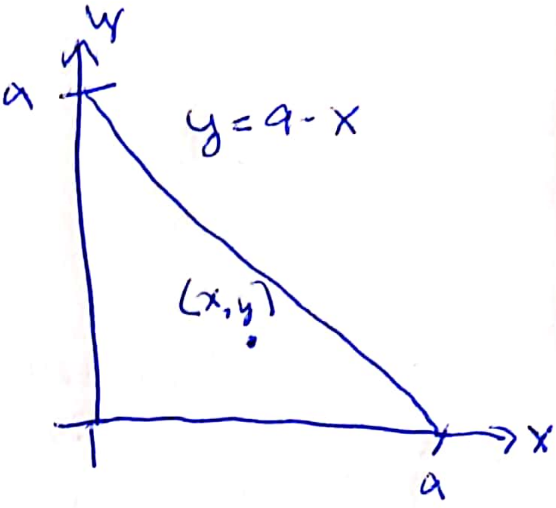
$$=\int_0^1\left[\frac{3}{2}xy^2+\frac{3}{2}x^2y^2+xy^3\right]_0^{\sqrt{x}}dx$$

$$=\int_0^1 \frac{3}{2}x^2+\frac{3}{2}x^3+x^{5/2}\,dx$$

$$=\left[\frac{1}{2}x^3+\frac{3}{8}x^4+\frac{2}{7}x^{7/2}\right]_0^1$$

$$=\frac{1}{2}+\frac{3}{8}+\frac{2}{7}=\frac{65}{56}\ \checkmark$$

(1.2) Find the center of mass of a lamina in the shape of an isosceles right triangle with equal sides of length a if the density at any point is proportional to the square of the distance from the vertex opposite the hypotenuse.



$$m = \int_0^a \int_0^{a-x} k(x^2+y^2)\,dy\,dx$$

$$= k\int_0^a \left[x^2 y + \frac{1}{3}y^3\right]_0^{a-x} dx$$

$$= k\int_0^a \left[x^2(a-x) + \frac{1}{3}(a-x)^3\right]dx$$

$$= k\int_0^a \left[a x^2 - x^3 + \frac{a^3}{3} - a^2 x + a x^2 - \frac{1}{3}x^3\right]dx$$

$$\rho(x,y) = k\left(\sqrt{x^2+y^2}\right)^2 = k(x^2+y^2)$$

$$= k\int_0^a \left[2ax^2 - \frac{4}{3}x^3 + \frac{a^3}{3} - a^2 x\right]dx = k\left(\frac{2}{3}a^4 - \frac{1}{3}a^4 + \frac{1}{3}a^4 - \frac{1}{2}a^4\right) = \frac{k}{6}a^4$$

$$M_y = k\int_0^a \int_0^{a-x} x(x^2+y^2)\,dy\,dx$$

$$= k\int_0^a \underbrace{\left[x^3(a-x) + \frac{1}{3}x(a-x)^3\right]}_{ax^3 - x^4 + \frac{1}{3}xa^3 - x^2a^2 + x^3 a - \frac{1}{3}x^4} dx$$

$$= k\int_0^a \left[2ax^3 - \frac{4}{3}x^4 + \frac{1}{3}a^3 x - x^2 a^2\right]dx$$

$$= k\left(\frac{2}{4}a^5 - \frac{4}{15}a^5 + \frac{1}{6}a^5 - \frac{1}{3}a^5\right) = \frac{k}{15}a^5$$

$$\bar{x} = \frac{M_y}{m} = \frac{\frac{k}{15}a^5}{\frac{k}{6}a^4} = \frac{2}{5}a.$$

by symmetry $\bar{y} = \frac{2}{5}a$ as well and $(\bar{x}, \bar{y}) = \left(\frac{2}{5}a, \frac{2}{5}a\right)$ ✓

(1.3) Suppose X and Y are random variables with joint density function.

$$f(X,Y)=\begin{cases}0.1e^{-(0.5X+0.2Y)} & \text{if } X\ge 0, Y\ge 0\\ 0 & \text{otherwise}\end{cases}$$

- Explain how to show f is a joint density function.
- Set-up an integral that represents $P(Y\ge 5)$.
- Find the expected values for X and Y.

we show f is a joint density fct by demonstrating that $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1.$

$$P(Y\geq 5) = \int_{-\infty}^{\infty}\int_{5}^{\infty} f(x,y)\,dy\,dx$$

either is okay.

$$= \int_{0}^{\infty}\int_{5}^{\infty} 0.1e^{-(0.5x+0.2y)}\,dy\,dx$$

$$\mu_1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x f(x,y)\,dy\,dx$$

$$= \int_{0}^{\infty}\int_{0}^{\infty} 0.1 x e^{-(0.5x+0.2y)}\,dy\,dx$$

$$= \int_{0}^{\infty} \lim_{t\to\infty}\left[\frac{0.1}{-0.2} x e^{-0.5x-0.2y}\right]_0^t dx$$

$$= -\frac{1}{2}\int_0^{\infty} x e^{-0.5x} \underbrace{\lim_{t\to\infty}\left(e^{-0.2t}-1\right)}_{-1} dx$$

$$= \frac{1}{2}\int_0^{\infty} x e^{-0.5x}\,dx$$

Let

$u = x \qquad dv = e^{-0.5x}\,dx$

$du = dx \qquad v = -\frac{1}{0.5}e^{-0.5x}$

$$= \frac{1}{2} \lim_{t\to\infty} \left[-2xe^{-0.5x} + 2\underbrace{\int e^{-0.5x} \, dx}_{-2e^{-0.5x}} \right]_0^t$$

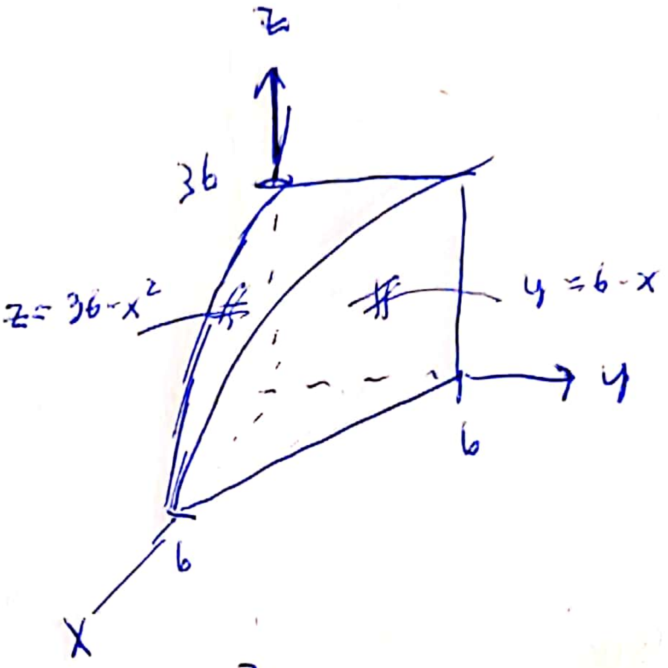
$$= \frac{1}{2} \lim_{t\to\infty} \left[\left(\frac{-2t}{e^{0.5t}} - \cancel{4e^{-0.5t}} \right) - (0 - 4) \right]$$

$$\overset{H}{=} \frac{1}{2} \lim_{t\to\infty} \left(4 - \cancel{\frac{2}{0.5e^{t}}} \right)$$

$$= 2$$

similarly, $\mu_2 = \frac{1}{2} \cdot 2 \cdot 5 = 5$

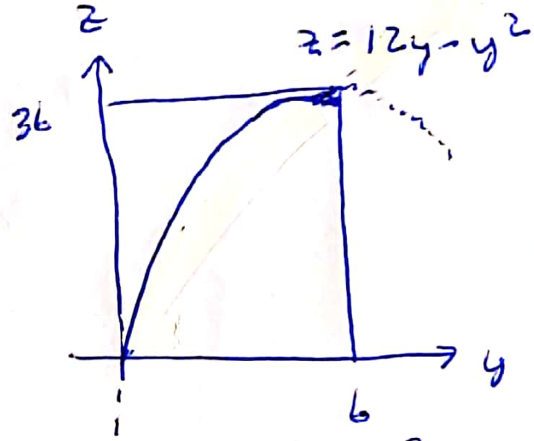
(1.4) Consider $I = \int_0^6 \int_0^{36-x^2} \int_0^{6-x} f(x,y,z)\,dy\,dz\,dx$. Rewrite this integral as an equivalent iterated integral (or integrals) in the five other orders.



$$I = \int_0^{36}\int_0^{\sqrt{36-z}}\int_0^{6-x} f\,dy\,dx\,dz$$

$$= \int_0^{6}\int_0^{6-y}\int_0^{36-x^2} f\,dz\,dx\,dy$$

$$= \int_0^{6}\int_0^{6-x}\int_0^{36-x^2} f\,dz\,dy\,dx$$



$$= \int_0^{6}\int_0^{12y-y^2}\int_0^{6-y} f\,dx\,dz\,dy$$

$$+ \int_0^{6}\int_{12y-y^2}^{36}\int_0^{\sqrt{36-z}} f\,dx\,dz\,dy$$

$$z = 36-(6-y)^2$$
$$= \cancel{36} - \cancel{36} + 12y - y^2$$
$$= 12y - y^2$$

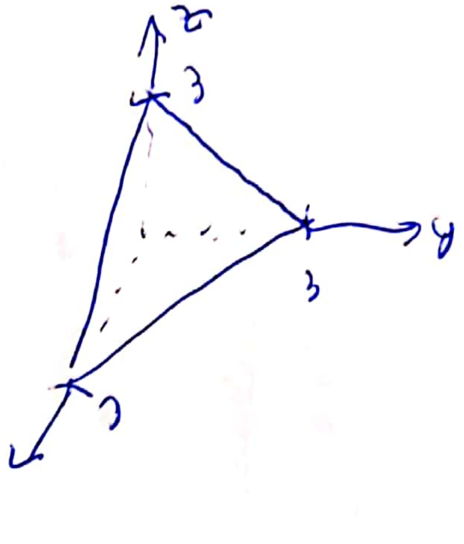
$$= \int_0^{36}\int_{6-\sqrt{36-z}}^{6}\int_0^{6-y} f\,dx\,dy\,dz$$

$$+ \int_0^{36}\int_0^{6-\sqrt{36-z}}\int_0^{\sqrt{36-z}} f\,dx\,dy\,dz \quad \checkmark$$

OR

$$z - 36 = -(6-y)^2$$
$$\Rightarrow 36 - z = (6-y)^2$$
$$\Rightarrow \pm\sqrt{36-z} = 6-y$$
$$\Rightarrow y = 6 \mp \sqrt{36-z}$$

(1.5) Find the centroid of the tetrahedron bounded by the planes $x=0,\ y=0,\ z=0$ and $x+y+z=3$ with density function $\rho(x,y,z)=10y$



$$m = \int_0^3 \int_0^{3-x} \underbrace{\int_0^{3-x-y} 10\,y\,dz}_{10y(3-x-y)}\,dy\,dx$$

$$= 10\int_0^3 \underbrace{\int_0^{3-x} (3y - xy - y^2)\,dy}_{\left[\frac{3}{2}y^2 - \frac{x}{2}y^2 - \frac{1}{3}y^3\right]_0^{3-x}}\,dx$$

$$= 10\int_0^3 \left[\frac{3}{2}(3-x)^2 - \frac{x}{2}(3-x)^2 - \frac{1}{3}(3-x)^3\right] dx$$

Let $u = 3-x$
$-du = dx$

$$= -10\int_3^0 \frac{3}{2}u^2 - \frac{3-u}{2}u^2 - \frac{1}{3}u^3\,du$$

$$= 10\int_0^3 \frac{u^3}{6}\,du = 10\left[\frac{u^4}{24}\right]_0^3 \quad \frac{810}{24} = \frac{135}{4}$$

$$M_{xz} = \int_0^3 \int_0^{3-x} \int_0^{3-x-y} 10y^2\,dz\,dy\,dx$$

$$= 10\int_0^3 \underbrace{\int_0^{3-x} (\underbrace{3y^2 - xy^2}_{(3-x)y^2} - y^3)\,dy}_{\left[(3-x)\frac{y^3}{3} - \frac{y^4}{4}\right]_0^{3-x}}\,dx$$

$$= 10\int_0^3 \left[\frac{(3-x)^4}{3} - \frac{(3-x)^4}{4}\right] dx$$

$$= \frac{5}{6}\int_0^3 (3-x)^4\, dx$$

Let $u = 3-x$

$-du = dx$

$$= \frac{5}{6}\int_0^3 u^4\, du = \left[\frac{\cancel{5}}{6}\cdot\frac{1}{\cancel{5}}u^5\right]_0^3 \quad \frac{243}{6} = \frac{81}{2}$$

so $\bar{y} = \dfrac{M_{xz}}{M} = \dfrac{81/2}{135/4} = \dfrac{6}{5}$

Since $p(x,y,z)$ includes only the variable y and the tetrahedron is symmetric in the x,y,z directions, $M_{xy} = M_{yz}$

$$M_{yz} = \int_0^3\int_0^{3-x}\underbrace{\int_0^{3-x-y} 10xy\, dz}_{10xy(3-x-y)}\, dy\, dx$$

$$= 10\int_0^3 \underbrace{\int_0^{3-x} 3xy - x^2y - xy^2\, dy}_{\left[\frac{3}{2}xy^2 - \frac{1}{2}x^2y^2 - \frac{1}{3}xy^3\right]_0^{3-x}}\, dx$$

$$= 10\int_0^3 \left[\underbrace{\frac{3}{2}x(3-x)^2 - \frac{1}{2}x^2(3-x)^2}_{\frac{1}{2}x(3-x)^3} - \frac{1}{3}x(3-x)^3\right] dx$$

$$= 10 \int_0^3 \frac{1}{6} x (3-x)^3 \, dx$$

Let $u = 3 - x$
$-du = dx$

$$= \frac{10}{6} \int_0^3 (3-u) u^3 \, du$$

$$= \frac{5}{3} \int_0^3 (3u^3 - u^4) \, du$$

$$= \frac{5}{3} \left[\frac{3}{4} u^4 - \frac{1}{5} u^5 \right]_0^3$$

$\frac{5}{3} \left(\frac{243}{4} - \frac{243}{5} \right) = \frac{81}{4}$

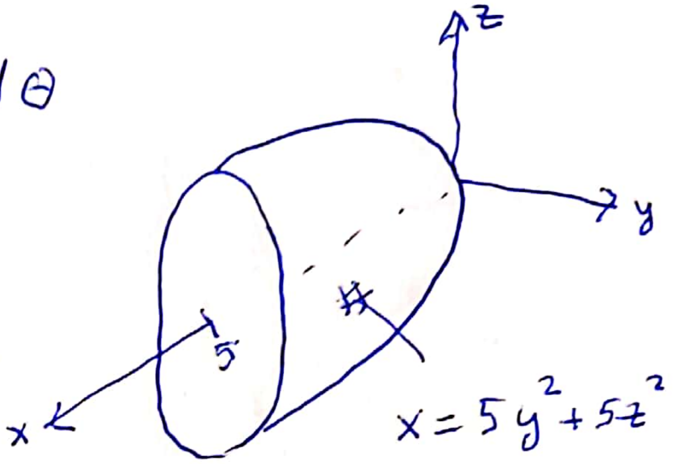
so $\bar{x} = \frac{M_{yz}}{m} = \frac{81/4}{135/4} = \frac{3}{5}$

by symmetry $\bar{z} = \frac{3}{5}$

so $(\bar{x}, \bar{y}, \bar{z}) = \left(\frac{3}{5}, \frac{6}{5}, \frac{3}{5} \right)$ ✓

(1.6) Evaluate the triple integral $I = \iiint_E 10x\,dV$ where E is bounded by $x = 5y^2 + 5z^2$ and $x = 5$.

$$I = \int_0^{2\pi}\int_0^{1}\int_{5y^2+5z^2 = 5r^2}^{5} 10x\, r\, dx\, dr\, d\theta$$



$$= 2\pi \int_0^1 \left[5x^2 r\right]_{5r^2}^{5} dr$$

$$= 2\pi \int_0^1 \left[125r - 125r^5\right] dr$$ use cylindrical in x, r, θ

$$= 250\pi \left[\frac{r^2}{2} - \frac{r^6}{6}\right]_0^1$$

$$= \frac{250\pi}{3}$$

(1.7) Find the volume of the solid that is enclosed by $z=\sqrt{x^2+y^2}$ and $x^2+y^2+z^2=18$.

$$V=\int_0^{2\pi}\int_0^{\pi/4}\int_0^{3\sqrt{2}} 1\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

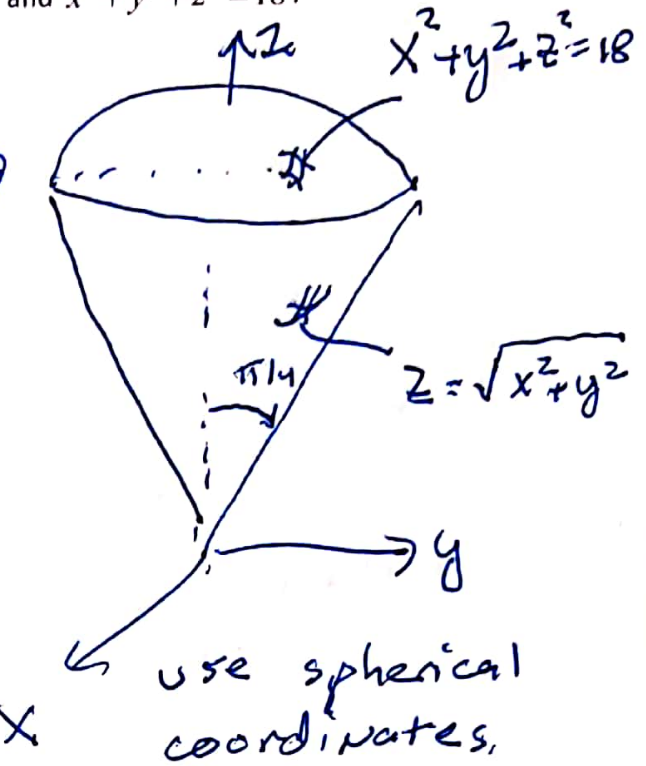
$$=2\pi\int_0^{3\sqrt{2}}\rho^2\,d\rho\int_0^{\pi/4}\sin\phi\,d\phi$$

$$=2\pi\left[\frac{\rho^3}{3}\right]_0^{3\sqrt{2}}\left[-\cos\phi\right]_0^{\pi/4}$$

$$=2\pi\left(\frac{54\sqrt{2}}{3}\right)\left(-\frac{\sqrt{2}}{2}+1\right)$$

$$=36\pi\sqrt{2}\left(1-\frac{\sqrt{2}}{2}\right)$$

$$=36\pi(\sqrt{2}-1)$$



use spherical coordinates.

$$V=\int_0^{2\pi}\int_0^{3}\int_r^{\sqrt{18-r^2}} 1\,r\,dz\,dr\,d\theta$$

$$=2\pi\int_0^3 r\sqrt{18-r^2}-r^2\,dr$$

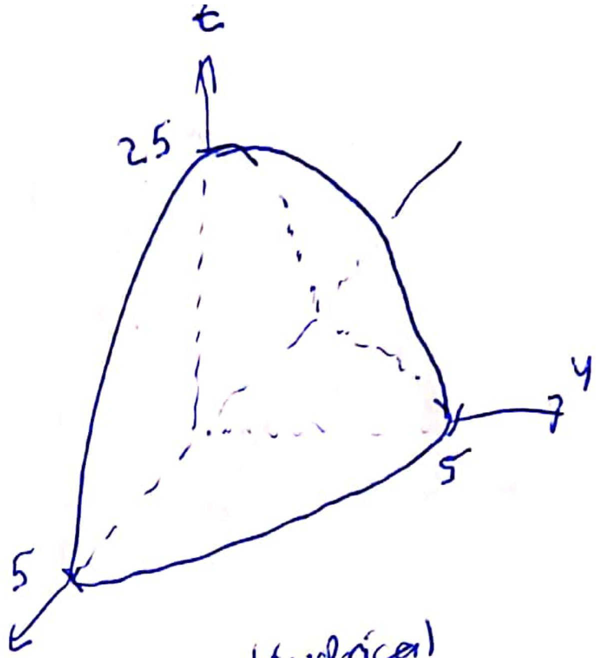
$$=2\pi\left[-\frac{1}{3}(18-r^2)^{3/2}-\frac{r^3}{3}\right]_0^3$$

$$=-\frac{2}{3}\pi\left(27-54\sqrt{2}+27\right)$$

$$=2\pi\left(18\sqrt{2}-18\right)$$

$$=36\pi(\sqrt{2}-1)$$

(1.8) Evaluate $I = \int_{-5}^{5}\int_{0}^{\sqrt{25-x^2}}\int_{0}^{25-x^2-y^2} \sqrt{x^2+y^2}\, dz\, dy\, dx$



use cylindrical coordinates

$$= \int_0^5 \int_0^{\pi} \int_0^{25-r^2} \sqrt{r^2}\, r\, dz\, d\theta\, dr$$

$$= \pi \int_0^5 \int_0^{25-r^2} r^2\, dz\, dr$$

$$= \pi \int_0^5 \underbrace{r^2 (25-r^2)}_{25r^2 - r^4}\, dr$$

$$= \pi \left[\frac{25}{3} r^3 - \frac{1}{5} r^5 \right]_0^5$$

$$= \frac{2}{15} \pi \cdot 5^5$$

$$= \frac{1250\pi}{3}$$

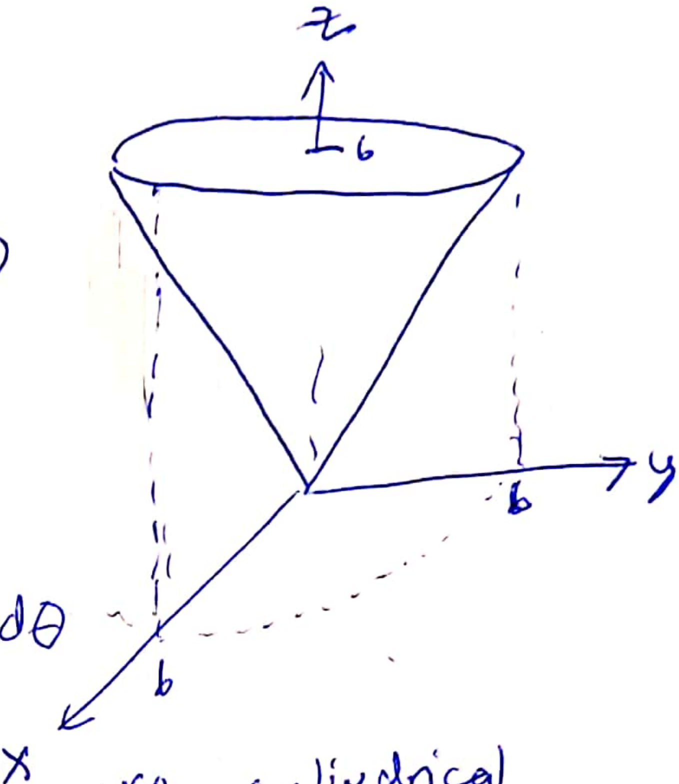
(1.9) Evaluate $I = \int_{-6}^{6}\int_{-\sqrt{36-y^2}}^{\sqrt{36-y^2}}\int_{\sqrt{x^2+y^2}}^{6} xz\,dz\,dx\,dy$

$$I = \int_0^{2\pi}\int_0^{6}\underbrace{\int_r^{6} rz\cos\theta \cdot r\,dz}_{\left[\frac{1}{2}r^2z^2\cos\theta\right]_r^6}\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{6}\frac{1}{2}r^2\cos\theta\,(36-r^2)\,dr\,d\theta$$

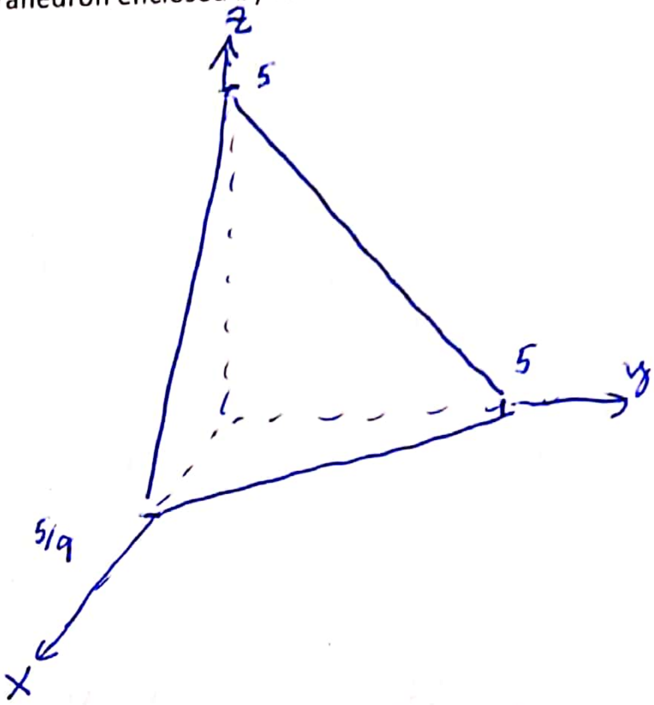
$$= \underbrace{\int_0^{2\pi}\cos\theta\,d\theta}_{0}\int_0^{6} r^2(36-r^2)\,dr$$

$$= ☺$$



use cylindrical coordinates.

(1.10) Use the methods of this class to find the volume of the tetrahedron enclosed by the coordinate planes and the plane $9x+y+z=5$



$$V = \int_0^5 \int_0^{5-z} \int_0^{(5-y-z)/9} 1 \, dx\, dy\, dz$$

$$= \frac{1}{9} \int_0^5 \underbrace{\int_0^{5-z} (5-y-z)\, dy}_{\left[5y - \frac{y^2}{2} - zy\right]_0^{5-z}} \, dz$$

$$= \frac{1}{9} \int_0^5 \underbrace{\left[5(5-z) - \frac{(5-z)^2}{2} - z(5-z)\right]}_{(5-z)^2 - \frac{(5-z)^2}{2}} dz$$

$$= \frac{1}{18} \int_0^5 \frac{(5-z)^2}{2}\, dz$$

let $u = 5-z$

$-du = dz$

$$= \frac{1}{18} \int_0^5 u^2 \, du$$

$$= \frac{1}{54} \left[u^3\right]_0^5$$

$$= \frac{125}{54}$$