Name: kay.

Assessment 2

Math& 264: Multivariable Calculus

Instructions: Please carefully complete these questions by hand. Be sure to show all work including sketching relevant graphs and providing exact answers.

Should you choose to work these on scratch paper, please do not put more than one question on a page. Additional sheets of paper are acceptable. WRITE YOUR NAME ON EVERY PAGE

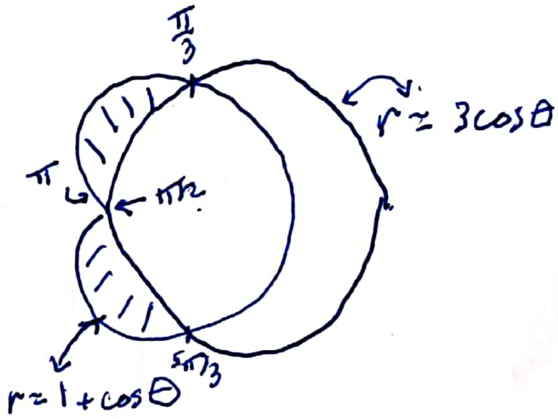
Upload your solutions to Gradescope by 9 am on Monday (10/12). During your presentation time, you will be asked to explain your thought process and reasoning on one randomly assigned question. Late solutions are available thru 2 pm with a 5% penalty (smaller penalties if you can document that this is a revision and only minor changes).

Please make sure to sign up for your presentation slot. If you are unavailable for any of the times available, please send me a note in Slack and we will find a time that works for you.

https://docs.google.com/spreadsheets/d/17eCs-TpdpqMTu97_csL7NoFjqaO1Qqaj0lxjtMOqYM4/edit?usp=sharing

(1.1) Use ~~a~~ double integral(s) to find the area of the region inside the cardioid $r = 1+\cos\theta$ and outside the circle $r = 3\cos\theta$.

$$\text{Half Area} = \int_{\frac{\pi}{3}}^{\pi}\int_0^{r=1+\cos\theta} 1\, r\, dr\, d\theta - \int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\int_0^{3\cos\theta} 1\, r\, dr\, d\theta$$



$$= \int_{\frac{\pi}{3}}^{\pi}\left[\frac{1}{2}r^2\right]_0^{1+\cos\theta} d\theta - \int_{\frac{\pi}{3}}^{\pi/2}\left[\frac{1}{2}r^2\right]_0^{3\cos\theta} d\theta$$

$$= \frac{1}{2}\int_{\frac{\pi}{3}}^{\pi} 1+2\cos\theta+\cos^2\theta\, d\theta - \frac{1}{2}\int_{\frac{\pi}{3}}^{\pi/2} 9\cos^2\theta\, d\theta$$

Solve

$1+\cos\theta = 3\cos\theta$

$\Rightarrow 1 = 2\cos\theta$

$$= \frac{1}{2}\int_{\frac{\pi}{3}}^{\pi} 1+2\cos\theta+\frac{1+\cos 2\theta}{2}\, d\theta - \frac{9}{4}\int_{\frac{\pi}{3}}^{\pi/2} 1+\cos 2\theta\, d\theta$$

$\Rightarrow \frac{1}{2} = \cos\theta$

$\Rightarrow \theta = \pm\frac{\pi}{3}$

$$= \frac{1}{2}\left[\frac{3}{2}\theta + 2\sin\theta + \frac{1}{4}\sin 2\theta\right]_{\frac{\pi}{3}}^{\pi} - \frac{9}{4}\left[\theta + \frac{1}{2}\sin 2\theta\right]_{\frac{\pi}{3}}^{\pi/2}$$

$$\boxed{\text{so Area} = \frac{\pi}{4}}$$

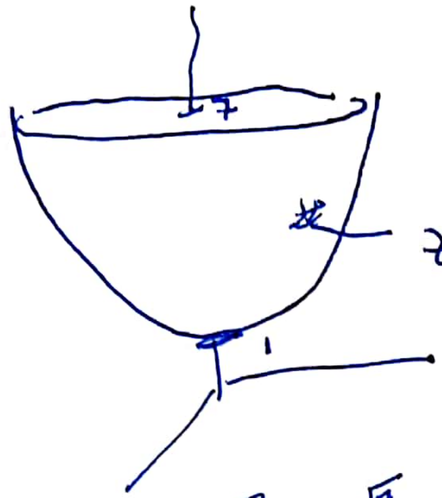
$$= \frac{1}{2}\left[\left(\frac{3\pi}{2}+0+0\right) - \left(\frac{\pi}{2}+\sqrt{3}+\frac{\sqrt{3}}{8}\right)\right] - \frac{9}{4}\left[\left(\frac{\pi}{2}+0\right) - \left(\frac{\pi}{3}+\frac{\sqrt{3}}{4}\right)\right] = \frac{\pi}{8}$$

(1.2) Use the methods developed in this course to find the volume of the region bounded by the paraboloid $z = 1 + 2x^2 + 2y^2$ and the plane $z = 7$ in the first octant.

solve $7 = 1 + 2r^2$

$6 = 2r^2$

$3 = r^2$



$z = 1 + 2x^2 + 2y^2$

$= 1 + 2r^2$

$$V = \int_0^{\frac{\pi}{2}} \int_0^{\sqrt{3}} [7 - (1 + 2r^2)]\, r \, dr \, d\theta$$

$$= \frac{\pi}{2} \int_0^{\sqrt{3}} (6r - 2r^3)\, dr$$

$$= \frac{\pi}{2} \left[3r^2 - \frac{1}{2} r^4 \right]_0^{\sqrt{3}}$$

$$= \frac{\pi}{2} \left(9 - \frac{9}{2} \right)$$

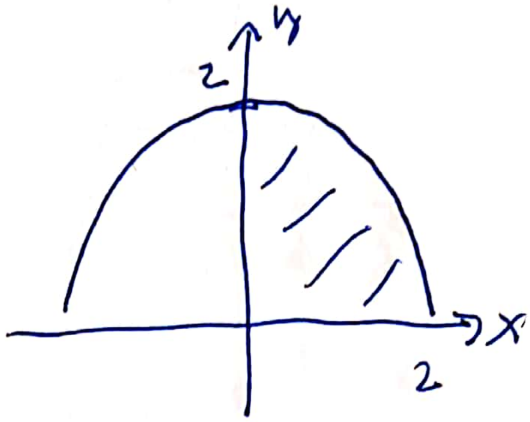
$$= \frac{9\pi}{4}$$

(1.3) Evaluate the iterated integral $I = \int_0^2 \int_0^{\sqrt{4-x^2}} e^{-x^2-y^2}\, dy\, dx$

$$I = \int_0^{\frac{\pi}{2}} \int_0^2 e^{-r^2}\, r\, dr\, d\theta$$

$$= \frac{\pi}{2} \left[-\frac{1}{2} e^{-r^2} \right]_0^2$$

$$= -\frac{\pi}{4} \left(e^{-4} - 1 \right)$$



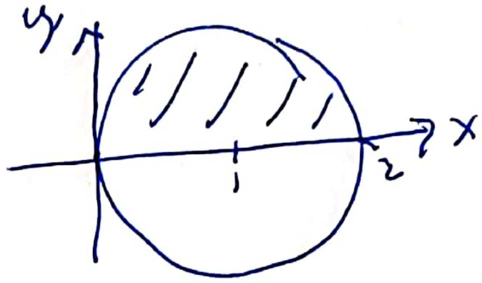
(1.4) Evaluate $I = \int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}\, dy\, dx$

$$y = \sqrt{2x - x^2}$$

$$= \sqrt{-(x^2-2x+1)+1}$$

$$= \sqrt{1-(x-1)^2}$$

$$\Rightarrow y^2 + (x-1)^2 = 1$$



$$\Rightarrow r^2 \sin^2\theta + (r\cos\theta - 1)^2 = 1$$

$$\Rightarrow r^2 \sin^2\theta + r^2\cos^2\theta - 2r\cos\theta = 0$$

$$\Rightarrow r^2 = 2r\cos\theta$$

$$\Rightarrow r = 2\cos\theta$$

$$\Rightarrow I = \int_0^{\pi/2}\int_0^{2\cos\theta} \sqrt{r^2}\, r\, dr\, d\theta$$

$$= \int_0^{\pi/2} \underbrace{\int_0^{2\cos\theta} r^2\, dr}_{\frac{1}{3}\left[r^3\right]_0^{2\cos\theta}}\, d\theta$$

$$= \frac{8}{3}\int_0^{\pi/2} \cos^3\theta\, d\theta$$

$\cos\theta(1-\sin^2\theta)$

$$= \frac{8}{3}\int_0^{\pi/2} \cos\theta - \sin^2\theta\cos\theta\, d\theta$$

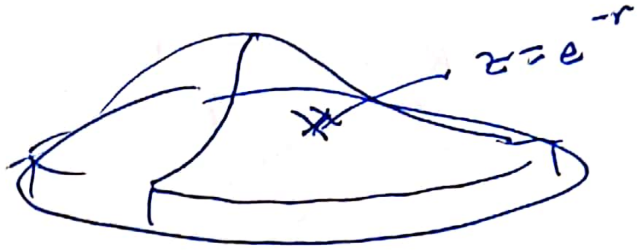
$$= \frac{8}{3}\left[+\sin\theta - \frac{1}{3}\sin^3\theta\right]_0^{\pi/2}$$

$$= \frac{8}{3}\left[+1 - \frac{1}{3}\right]$$

$$= \frac{16}{9}$$

(1.5) An agricultural sprinkler distributes water in a circular pattern of radius 100 ft. It supplies water to a depth of e^{-r} feet per hour at a distance of r feet from the sprinkler. Determine an expression for the average amount of water per hour per square foot supplied to the region ~~inside the circle~~.

$$V = \int_0^{2\pi}\int_0^{100} 104\, r e^{-r}\, dr\, d\theta$$



$$= 104 \int_0^{2\pi} \left[-re^{-r} + \int e^{-r}\, dr \right]_0^{100} d\theta \qquad u = r \qquad dv = e^{-r}\, dr$$

$$du = dr \qquad v = -e^{-r}$$

$$= 208\pi \left[-re^{-r} - e^{-r} \right]_0^{100}$$

$$= 208\pi \left[\left(-100e^{-100} - e^{-100}\right) - (0 - 1) \right]$$

$$= 208\pi \left(1 - 101e^{-100}\right)$$

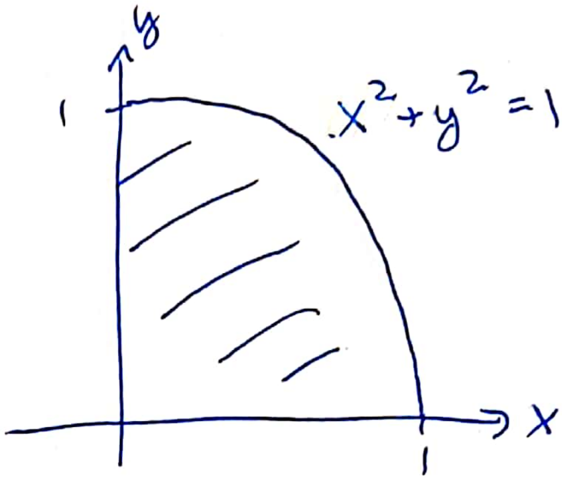
$$V_{AVE} = \frac{208\cancel{\pi}\left(1 - 101e^{-100}\right)}{\cancel{\pi}(100)^2}$$

$$= 104 \cdot \frac{1 - 101e^{-100}}{5000}$$

this works out to about 0.25 in of water / sqft.

... this sounds good until you notice that the water is 104 ft deep at the sprinkler!

(1.6) A lamina occupies the part of the disk $x^2 + y^2 \leq 1$ in the first quadrant. Find its center of mass if the density at any point is proportional to its distance from the x-axis.



$\rho(x,y) = ky$

$$M = \int_0^{\pi/2}\int_0^1 k r \sin\theta \cdot r \, dr \, d\theta$$

$$= k \underbrace{\int_0^{\frac{\pi}{2}} \sin\theta \, d\theta}_{1} \underbrace{\int_0^1 r^2 \, dr}_{\frac{1}{3}}$$

$$= \frac{k}{3}$$

$$M_x = \int_0^{\frac{\pi}{2}}\int_0^1 k(r\sin\theta)^2 \, r \, dr \, d\theta$$

$$= k \int_0^{\frac{\pi}{2}} \sin^2\theta \, d\theta \int_0^1 r^3 \, dr$$

$$= \frac{k}{4}\int_0^{\pi/2} \frac{1-\cos 2\theta}{2} \, d\theta$$

$$= \frac{k}{8}\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\pi/2}$$

$$= \frac{k\pi}{16}$$

$$M_y = \int_0^{\pi/2}\int_0^1 k(r\cos\theta)(r\sin\theta)\, r\, dr\, d\theta$$

$$= k\int_0^{\pi/2} \sin\theta\cos\theta\, d\theta \int_0^1 r^3\, dr$$

$$= \frac{k}{4}\left[\frac{1}{2}\sin^2\theta\right]_0^{\pi/2}$$

$$= \frac{k}{8}$$

$$\bar{x} = \frac{M_y}{m} = \frac{k/8}{k/3} = \frac{3}{8}$$

$$\bar{y} = \frac{M_x}{m} = \frac{k\pi/16}{k/3} = \frac{3\pi}{16}$$

so $(\bar{x}, \bar{y}) = \left(\frac{3}{8}, \frac{3\pi}{16}\right)$