Name: Key

**Assessment 1**

**Math& 264: Multivariable Calculus**

Instructions: Please carefully complete these questions by hand. Be sure to show all work including sketching relevant graphs and providing exact answers.

Should you choose to work these on scratch paper, please do not put more than one question on a page. Additional sheets of paper are acceptable.

Upload your solutions to Gradescope by 9 am on Monday (10/5). During your presentation time, you will be asked to explain your thought process and reasoning on one randomly assigned question. Late solutions are available thru 2 pm with a 5% penalty (smaller penalties if you can document that this is a revision and only minor changes).

Please make sure to sign up for your presentation slot. If you are unavailable for any of the times available, please send me a note in Slack and we will find a time that works for you.

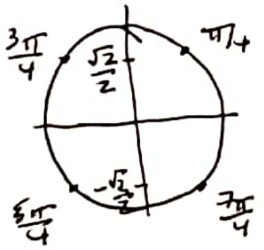
https://docs.google.com/spreadsheets/d/17eCs-TpdpqMTu97_csL7NoFjqaO1Qqaj0IxjtMOqYM4/edit?usp=sharing

(1.1) Find ALL values of $x$ that satisfy the equation $2\sin^2(x)=1$.

$\Rightarrow \sin^2 x = \frac{1}{2}$

$x = \frac{\pi}{4} + k\frac{\pi}{2}$ where $k \in \mathbb{Z}$.

$\Rightarrow \sin x = \pm\sqrt{\frac{1}{2}}$

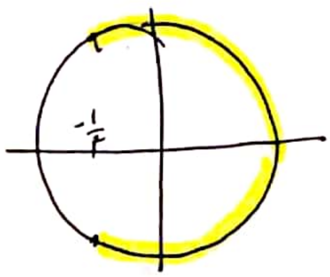



(1.2) Find all values if $x$ in the interval $[0,2\pi]$ that satisfy the inequality $2\cos(x)+1>0$

$\Rightarrow 2\cos x > -1$

$[0, \frac{2\pi}{3}) \cup (\frac{4\pi}{3}, 2\pi]$

$\Rightarrow \cos x > -\frac{1}{2}$



$I=$

(1.3) Evaluate the indefinite integral $\int \frac{dt}{\cos^2(t)\sqrt{1+\tan(t)}}$

Let $u = 1 + \tan t$

$du = \sec^2 t \, dt$

$$\Rightarrow I = \int \frac{du}{\sqrt{u}}$$

$$= 2\sqrt{u} + C$$

$$= 2\sqrt{1+\tan t} + C$$

$I=$

(1.4) Determine whether the integral $\int_3^\infty \frac{1}{(x-2)^{3/2}} dx$ is convergent or divergent (justify your answer).

Evaluate the integral if it is convergent.

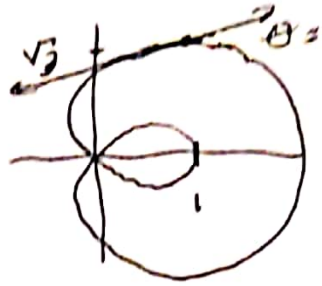
$$I = \lim_{t\to\infty} \int_3^t \frac{1}{(x-2)^{3/2}} dx$$

$$= \lim_{t\to\infty} \left[ -2(x-2)^{-1/2} \right]_3^t$$

$$= \lim_{t\to\infty} \left( \cancel{\frac{-2}{\sqrt{t-2}}}^{\,0} + \frac{2}{\sqrt{1}} \right)$$

$$= 2$$

The integral converges.




(1.5) Find the equation of the tangent line to $r = 1 + 2\cos\theta$ when $\theta = \pi/3$

$r = 1 + 2\cos\theta$

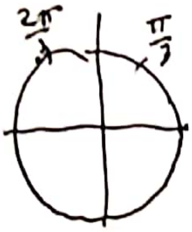
$\frac{dr}{d\theta} = -2\sin\theta$

$$\boxed{\text{Tangent Line} \quad y - \sqrt{3} = \frac{1}{3\sqrt{3}}(x-1)}$$

or $y = \frac{x}{3\sqrt{3}} + \frac{8}{3\sqrt{3}}$

And $\frac{dy}{dx} = \frac{-2\sin\theta(\sin\theta) + (1+2\cos\theta)\cos\theta}{-2\sin\theta(\cos\theta) - (1+2\cos\theta)\sin\theta}$

$= \frac{2\cos^2\theta + \cos\theta - 2\sin^2\theta}{-\sin\theta - 4\sin\theta\cos\theta}$




$= -\frac{2\cos 2\theta + \cos\theta}{\sin\theta + 2\sin 2\theta}\Big|_{\theta = \frac{\pi}{3}} \quad -\frac{2(-\frac{1}{2}) + \frac{1}{2}}{\frac{\sqrt{3}}{2} + 2\cdot\frac{\sqrt{3}}{2}} = \frac{1}{3\sqrt{3}}$

(1.6) Find the coordinates of the points on the curve $r = 1 - \sin\theta$ where the tangent line is horizontal or vertical.

$r = 1 - \sin\theta$

$\frac{dr}{d\theta} = -\cos\theta$

check $\frac{dy}{dx}$ @ $\theta = \frac{\pi}{2}$

$\lim_{\theta\to\frac{\pi}{2}} \frac{\cos\theta(1-2\sin\theta)}{(2\sin\theta+1)(\sin\theta-1)} = \underbrace{\lim_{\theta\to\frac{\pi}{2}} \frac{1-2\sin\theta}{2\sin\theta+1}}_{-1/3} \cdot \lim_{\theta\to\frac{\pi}{2}} \frac{\cos\theta}{\sin\theta - 1}$

(H) $= -\frac{1}{3}\lim_{\theta\to\frac{\pi}{2}} \frac{-\sin\theta}{\cos\theta}$ DNE (so vertical)

And $\frac{dy}{dx} = \frac{-\cos\theta(\sin\theta) + (1-\sin\theta)\cos\theta}{-\cos\theta(\cos\theta) - (1-\sin\theta)\sin\theta}$

$= \frac{\cos\theta - 2\sin\theta\cos\theta}{\sin^2\theta - \cos^2\theta - \sin\theta} \leftarrow \sin^2\theta - (1-\sin^2\theta) - \sin\theta$

$2\sin^2\theta - \sin\theta - 1$

$= \frac{\cos\theta(1-2\sin\theta)}{(2\sin\theta+1)(\sin\theta-1)}$

Horizontal: $\cos\theta = 0$ or $\sin\theta = \frac{1}{2}$

$\Rightarrow \theta \cancel{=} \frac{\pi}{2}$ or $\frac{3\pi}{2}$ or $\theta = \frac{\pi}{6}, \frac{5\pi}{6}$

$\Rightarrow (2, \frac{3\pi}{2})$

$(\frac{1}{2}, \frac{\pi}{6})$ and $(\frac{1}{2}, \frac{5\pi}{6})$

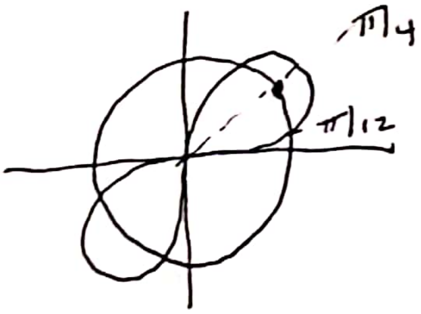
vertical: $\sin\theta = -\frac{1}{2}$ or $\sin\theta = 1$

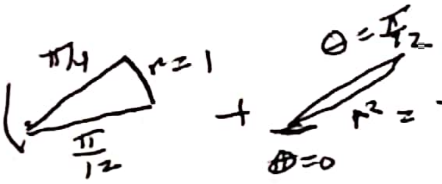
$\Rightarrow \theta = \frac{7\pi}{6}, \frac{11\pi}{6}$ or $\theta = \frac{\pi}{2}$

$\Rightarrow (0, \frac{\pi}{2})$

$(\frac{3}{2}, \frac{7\pi}{6})$ and $(\frac{3}{2}, \frac{11\pi}{6})$

(1.7) Find the area inside $r^2 = 2\sin 2\theta$ AND $r = 1$.




Area $= 4\Big($  $\Big)$


$$\text{Area} = 4\left[\int_{\pi/12}^{\pi/4} \frac{1}{2}(1)^2\, d\theta + \frac{1}{2}\int_0^{\pi/12} 2\sin 2\theta\, d\theta\right]$$

solve $1 = 2\sin 2\theta$

$\Rightarrow \frac{1}{2} = \sin 2\theta$

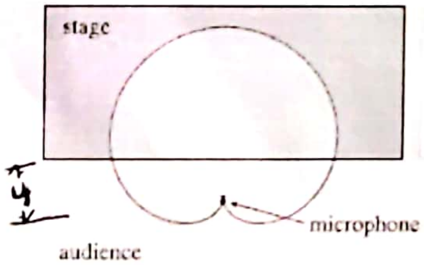
$\Rightarrow \theta = \frac{\pi}{12}$

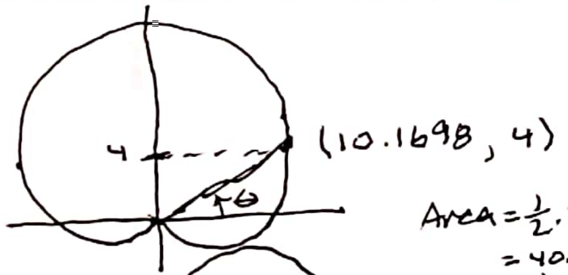
$$= 4\left(\frac{1}{2}\cdot\frac{\pi}{6} + \frac{1}{2}\Big[-\cos 2\theta\Big]_0^{\pi/12}\right)$$

$$= 4\left(\frac{\pi}{12} - \frac{\sqrt{3}}{4} + \frac{1}{2}\right)$$

$$= \frac{\pi}{3} - \sqrt{3} + 2$$

(1.8) When recording live performances, sound engineers often us a microphone with a cardioid pickup pattern because it suppresses noise from the audience. Suppose the microphone is placed 4 m from the front of the stage and the boundary of the optimal pickup region is given by the cardioid $r = 8 + 8\sin\theta$, where $r$ is measured in meters and the microphone is on a pole. The musicians want to know the area they will have on stage within the optimal pickup range of the microphone. Answer their questions.




Area $= \frac{1}{2}\cdot 2\cdot 10.1698\cdot 4$
$= 40.6794$

Area $=$  $-$ 

$y = r\sin\theta$

$= (8 + 8\sin\theta)\sin\theta$

solve $4 = 8(1 + \sin\theta)\sin\theta$

$\Rightarrow 0 = 2\sin^2\theta + 2\sin\theta - 1$

$$\Rightarrow \sin\theta = \frac{-2 \pm \sqrt{4 - 4(2)(-1)}}{4}$$

$$= \frac{-2 \pm 2\sqrt{3}}{4}$$

$\Rightarrow \theta \approx 0.3747$ & $\theta \approx \pi - 0.3747$

$$= 2\int_{0.3747}^{\pi/2} \frac{1}{2}(8 + 8\sin\theta)^2\, d\theta - 40.6794$$

$$= \int_{0.3747}^{\pi/2} 64 + 128\sin\theta + 64\sin^2\theta\, d\theta - 40.6794$$

$$= 64\int_{0.3747}^{\pi/2} 1 + 2\sin\theta + \frac{1 - \cos 2\theta}{2}\, d\theta - 40.6794$$

$\hookrightarrow$ $\text{Area} = 64\left[\frac{3}{2}\theta - 2\cos\theta - \frac{\sin 2\theta}{4}\right]_{0.3747}^{\pi/2} - 40.6794$

$= 96\left(\frac{\pi}{2} - 0.3747\right) + 128\cos(0.3747) + 16\sin(2 * 0.3747)$

$- 40.6794$

$= 204.16 m^2$

(1.9) Find the exact length of the polar curve $r=\theta^2$ on $0\le\theta\le \pi/2$.

$$L=\int_0^{\frac{\pi}{2}}\sqrt{\theta^4+(2\theta)^2}\,d\theta$$

$$=\int_0^{\pi/2}\theta\sqrt{\theta^2+4}\,d\theta$$

Let $u=\theta^2+4$

$$\frac{du}{2}=\theta\,d\theta$$

$$u(0)=4\ ;\ u\left(\tfrac{\pi}{2}\right)=\frac{\pi^2}{4}+4$$

$$\Rightarrow L=\frac{1}{2}\int_4^{4+\frac{\pi^2}{4}}\sqrt{u}\,du$$

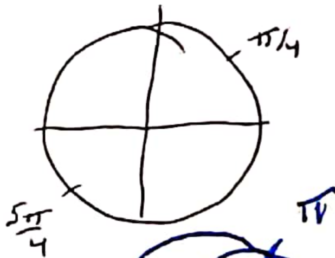
$$\Rightarrow\ =\frac{1}{2}\left[\frac{2}{3}u^{3/2}\right]_4^{4+\frac{\pi^2}{4}}$$

$$=\frac{1}{3}\left(\left(4+\frac{\pi^2}{4}\right)^{3/2}-8\right)$$

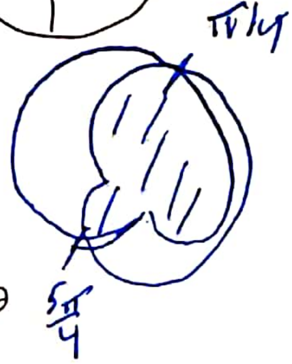
OR

$$=\frac{(\pi^2+16)^{3/2}}{24}-\frac{8}{3}$$

(1.10) Find the area inside $r=5+4\sin\theta$ AND $r=5+4\cos\theta$.




$$\text{Area}=2\int_{\frac{\pi}{4}}^{\frac{5\pi}{4}}\frac{1}{2}(5+4\cos\theta)^2\,d\theta$$




$$=\int_{\frac{\pi}{4}}^{\frac{5\pi}{4}}(25+40\cos\theta+16\cos^2\theta)\,d\theta$$

$$=25\pi+40\Big[\sin\theta\Big]_{\frac{\pi}{4}}^{\frac{5\pi}{4}}+8\int_{\frac{\pi}{4}}^{\frac{5\pi}{4}}(1+\cos 2\theta)\,d\theta$$

$$=25\pi+40\left(-2\cdot\frac{\sqrt{2}}{2}\right)+8\left[\theta+\frac{1}{2}\sin 2\theta\right]_{\frac{\pi}{4}}^{\frac{5\pi}{4}}$$

$$=25\pi-40\sqrt{2}+8\pi+4\left(\sin\left(\tfrac{5\pi}{2}\right)-\sin\left(\tfrac{\pi}{2}\right)\right)$$

$$=33\pi-40\sqrt{2}$$