HW 46 min + 8 Qs.

Stewart 16.4

The preceding section gave a version of the Fundamental Theorem of Calculus that applies to line integrals. In this lesson and for the remainder of the course, you will see additional extensions of the Fundamental Theorem that apply to regions in $\mathbb{R}^2$ and $\mathbb{R}^3$. All these fundamental theorems share a common feature. Part 2 of the Fundamental Theorem of Calculus says

$$\int_a^b \frac{df}{dx}\,dx = f(b) - f(a)$$

which relates the integral of $\frac{df}{dx}$ on an interval $[a,b]$ to the values of f on the boundary of $[a,b]$. The Fundamental Theorem for line integrals says

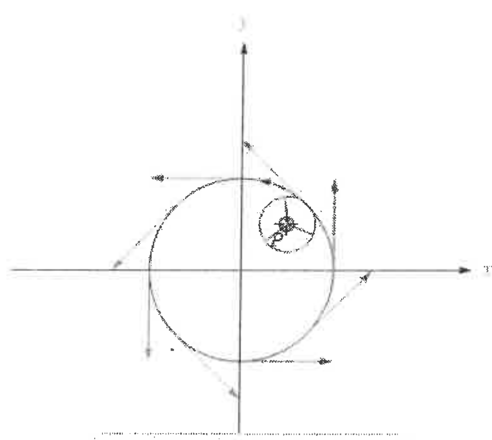
$$\int_C \nabla\varphi \cdot d\bar{r} = \varphi(B) - \varphi(A)$$

which relates the integral of $\nabla\varphi$ on a piecewise-smooth oriented curve C to the boundary values of φ. (The boundary consists of the two endpoints A and B.)

The subject of this section is Green's Theorem, which is another step in this progression. It relates the double integral of derivatives of a function over a region in $\mathbb{R}^2$ to function values on the boundary of that region.

I. Circulation form of Green's Theorem

Recall that the circulation $\oint_C \bar{F} \cdot d\bar{r} = \oint_C \bar{F} \cdot \bar{T}\,ds$ measures the net component of $\bar{F}$ in the direction tangential to C. A nonzero circulation on a closed curve says that the vector field must have some property inside the curve that produces the circulation. You can think of this property as a net rotation.



$\mathbf{F} = \langle -y, x\rangle$ positive (counterclockwise) circulation on C.

check out manipulate 17.31

Theorem: Green's Theorem (Circulation Form)

requires parameterization

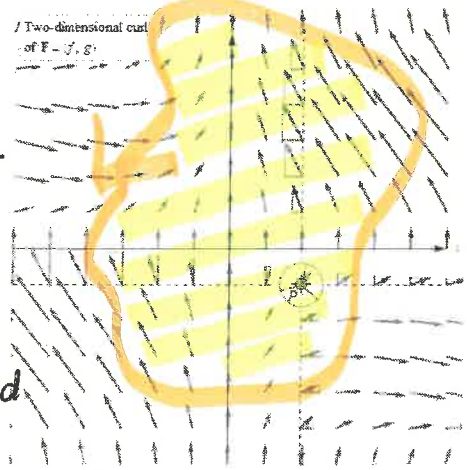
Let C be a simple closed piecewise-smooth curve, oriented counterclockwise, that encloses a connected and simply connected region R in the plane. Assume $\vec{F} = \langle f, g \rangle$, where f and g have continuous first partial derivatives in R. Then

no parameterization

work/circulation = $\oint_C \vec{F}\cdot d\vec{r} = \oint_C \vec{F}\cdot\vec{T}\,ds = \oint_C f\,dx + g\,dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA$

Interpretation: The integrand of the double integral is the two-dimensional curl of the vector field. So Green's Theorem says that the circulation around the boundary of a region is equal to net curl across the region.

circulation is about the tangential component of the field along C

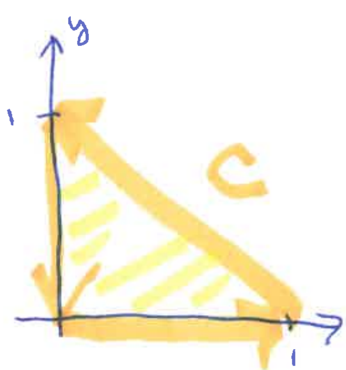


curl measures the rotation of the field at a point.

check out manipulate 17.32

work/circulation =

Ex 1: Evaluate $\oint_C x^4\,dx + xy\,dy$, where C is the triangular curve consisting of the line segments from (0,0) to (1,0), from (1,0) to (0,1), and from (0,1) to (0,0).



work $= \oint_C \vec{F}\cdot d\vec{r}$ with $\vec{F} = \langle x^4, xy \rangle$

$= \iint_{\triangle} (y - 0)\,dA$

$= \int_0^1 \int_{y=0}^{y=1-x} y\,dy\,dx$

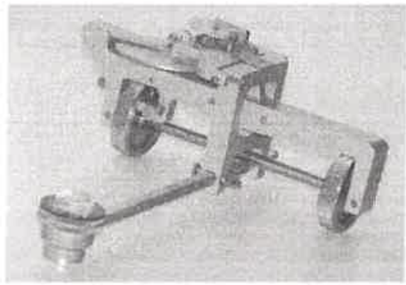
$= \int_0^1 \underbrace{\frac{1}{2}(1-x)^2}_{\frac{1}{2}(1-2x+x^2)}\,dx$

$= \frac{1}{2}\int_0^1 (1 - 2x + x^2)\,dx$

$= \frac{1}{2}\left[x - x^2 + \frac{1}{3}x^3\right]_0^1$

$= \frac{1}{6}$ Total work done.

An interesting application of Green's Theorem is that it can be used to find the area enclosed by a curve. In the 1850's (and later) this concept was used to develop a tool called a planimeter with which you could trace the boundary of a region (say on a map) and the device ~~would~~ tells you the area enclosed.



you can build one out of Legos!

The formula itself comes from applying Green's Theorem to the two fields $\vec{F} = \langle f, g \rangle = \langle 0, x \rangle$ and $\vec{F} = \langle f, g \rangle = \langle y, 0 \rangle$. Their difference gives: Area of R enclosed by $C = \frac{1}{2}\oint_C (x\,dy - y\,dx)$.

Ex 2: Use Green's Theorem to derive the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Area of the ellipse $= \iint_{\text{ellipse}} 1 \, dA$

$\vec{r}(t) = \langle a\cos t, b\sin t \rangle$ on $0 \le t \le 2\pi$

$= \frac{1}{2}\oint_{\text{ellipse}} x\,dy - y\,dx$

$\vec{r}\,'(t) = \langle -a\sin t, b\cos t \rangle$

$= \frac{1}{2}\int_0^{2\pi} [a\cos t \, b\cos t - b\sin t(-a\sin t)]\,dt$

$= \frac{ab}{2}\int_0^{2\pi} 1 \, dt$

$= ab\pi.$

II. Flux form of Green's Theorem

Recall that the outward flux of $\vec{F}$ across the closed curve C is $\oint_C \vec{F}\cdot\vec{n}\,ds = \oint_C (f\,dy - g\,dx)$.

Applying Green's Theorem, we have:

Theorem: Green's Theorem (Flux Form)

Let C be a simple closed piecewise-smooth curve, oriented counterclockwise, that encloses a connected and simply connected region R in the plane. Assume $\vec{F} = \langle f, g\rangle$, where f and g have continuous first partial derivatives in R. Then

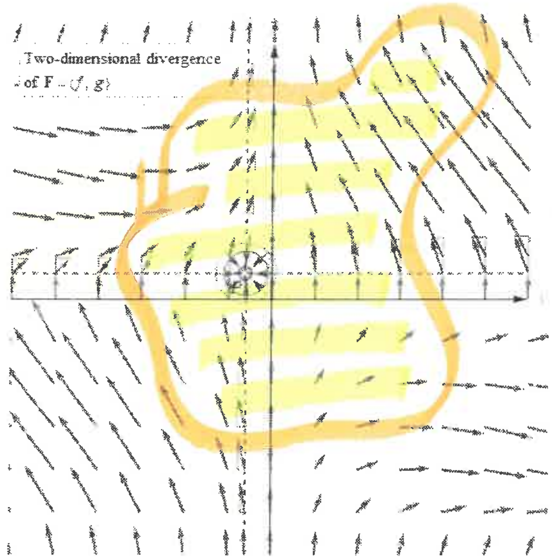
requires parameterization

no parameterization required

$$\text{flux} = \oint_C \vec{F}\cdot\vec{n}\,ds = \oint_C (f\,dy - g\,dx) = \iint_R \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}\right) dA$$

where $\vec{n}$ is the outward unit normal vector on the curve.

Interpretation: The two line integrals on the left side give the outward flux of the vector field across C. The double integral on the right side involves the quantity $\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}$, which is the property of the vector field that produces the flux across C. This factor is called the two-dimensional divergence. That is, the net flux across the boundary is equal to the total divergence across the enclosed region.



flux is about the normal component of the field along C.

divergence measures the amount the field attracts/repels at a point.

Flux =

Ex 3: Evaluate $\oint_C (2x-3y)\,dy - (3x+4y)\,dx$, where C is the unit circle oriented ccw.

$\uparrow$ f $\quad\quad$ $\uparrow$ g

$$\vec{F} = \langle 2x-3y,\ 3x+4y \rangle$$

$$\text{Flux} = \int_C \vec{F} \cdot \vec{n}\, ds$$

$$= \iint_{\text{unit circle}} 2+4\, dA$$

$$= 6 \cdot \pi (1)^2$$

$$= 6\pi.$$

III. Stream functions (a cool connection) and parallel properties

One reason for introducing two forms of Green's Theorem (circulation and flux) is that it will simplify later work with Stoke's Theorem and the Divergence Theorem. To complete this, we need to go backward and connect divergence and flux to a concept introduced at the beginning of the chapter: the stream function. The stream function is used in fluid dynamics (a branch of engineering) where it is used to model fluids that are incompressible (ex: hydraulics).

Let us begin by looking at the parallels:

$$\text{curl} \Leftrightarrow \text{circulation/work} \Leftrightarrow \underbrace{\text{conservative}}_{\text{curl}=0} \Leftrightarrow \underbrace{\text{potential function}}_{\varphi}$$

$$\text{divergence} \Leftrightarrow \text{flux} \Leftrightarrow \underbrace{\text{source - free}}_{\text{div}=0} \Leftrightarrow \underbrace{\text{stream function}}_{\psi}$$

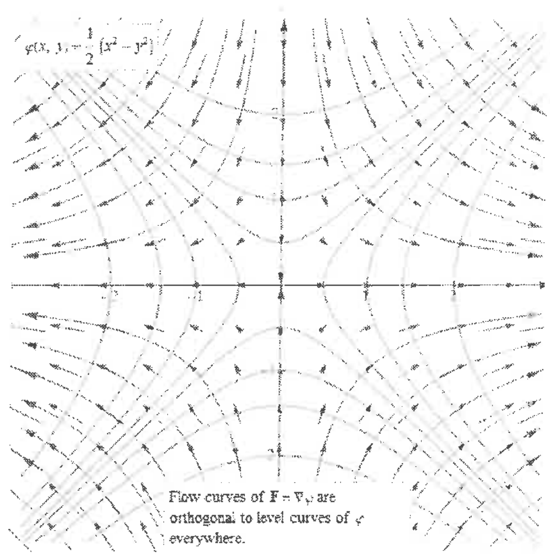
And the parallel processes:

Integrate $\dfrac{\partial \varphi}{\partial x} = f$ and $\dfrac{\partial \varphi}{\partial y} = g$ to find the potential function φ.

Integrate $\dfrac{\partial \psi}{\partial y} = f$ and $\dfrac{\partial \varphi}{\partial x} = -g$ to find the stream function ψ.

Vocabulary: If the stream function exists, we say that the field is source-free.

It can be shown that the vector field $\vec{F}$ is everywhere tangent to the streamlines, which means that a graph of the streamlines shows the flow of the vector field. Finally, just as circulation integrals of a conservative vector field are independent of path, flux integrals of a source-free field are also independent of path.



Here are the parallel properties of conservative and source-free vector fields in two dimensions. We assume C is a simple piecewise-smooth oriented curve and is either closed or has endpoints A and B.

Conservative Fields $\mathbf{F} = (f, g)$

- curl $= \dfrac{\partial g}{\partial x} - \dfrac{\partial f}{\partial y} = 0$
- Potential function φ with $\mathbf{F} = \nabla\varphi$ or $f = \dfrac{\partial \varphi}{\partial x}$, $g = \dfrac{\partial \varphi}{\partial y}$
- Circulation $= \oint_C \mathbf{F}\cdot d\mathbf{r} = 0$ on all closed curves C.
- Evaluation of line integral $\int_C \mathbf{F}\cdot d\mathbf{r} = \varphi(B) - \varphi(A)$

Source-Free Fields $\mathbf{F} = (f, g)$

- divergence $= \dfrac{\partial f}{\partial x} + \dfrac{\partial g}{\partial y} = 0$
- Stream function ψ with $f = \dfrac{\partial \psi}{\partial y}$, $g = -\dfrac{\partial \psi}{\partial x}$
- Flux $= \oint_C \mathbf{F}\cdot\mathbf{n}\, ds = 0$ on all closed curves C.
- Evaluation of line integral $\int_C \mathbf{F}\cdot\mathbf{n}\, ds = \psi(B) - \psi(A)$

With Green's Theorem in the picture, we may also give a concise summary of the various cases that arise with line integrals of both the circulation and flux types.

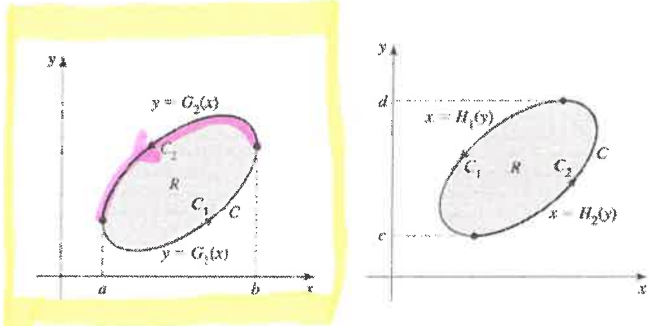
Circulation/work integrals: $\int_C \mathbf{F}\cdot\mathbf{T}\, ds = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_C f\, dx + g\, dy$

	C closed	C not closed
F conservative ($\mathbf{F} = \nabla\varphi$)	$\oint_C \mathbf{F}\cdot d\mathbf{r} = 0$	$\int_C \mathbf{F}\cdot d\mathbf{r} = \varphi(B) - \varphi(A)$
F not conservative	Green's Theorem $\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_R (g_x - f_y)\, dA$	Direct evaluation $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_a^b (f\, x' + g\, y')\, dt$

Flux integrals: $\int_C \mathbf{F}\cdot\mathbf{n}\, ds = \int_C f\, dy - g\, dx$

	C closed	C not closed
F source free ($f = \psi_y$, $g = -\psi_x$)	$\oint_C \mathbf{F}\cdot\mathbf{n}\, ds = 0$	$\int_C \mathbf{F}\cdot\mathbf{n}\, ds = \psi(B) - \psi(A)$
F not source free	Green's Theorem $\oint_C \mathbf{F}\cdot\mathbf{n}\, ds = \iint_R (f_x + g_y)\, dA$	Direct evaluation $\int_C \mathbf{F}\cdot\mathbf{n}\, ds = \int_a^b (f\, y' - g\, x')\, dt$

A proof of Green's Theorem when restricted to the special regions pictured:



$R=\{(x,y): a \le x \le b \text{ and } G_1(x) \le y \le G_2(x)\}$ and $R=\{(x,y): c \le y \le d \text{ and } H_1(y) \le x \le H_2(y)\}$

Here the circulation form of Green's Theorem is $\oint_C f\,dx + g\,dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA$

focus

$$\iint_R \frac{\partial f}{\partial y}\, dA = \int_a^b \int_{G_1(x)}^{G_2(x)} \frac{\partial f}{\partial y}\, dy\, dx$$

$$= \int_a^b \left[f(x, G_2(x)) - f(x, G_1(x)) \right] dx$$

$$= \int_{-C_2} f\,dx - \int_{C_1} f\,dx$$

$$= -\int_{C_2} f\,dx - \int_{C_1} f\,dx$$

$$= -\oint_C f\,dx$$

similarly $\iint_R \frac{\partial g}{\partial x}\, dA = \oint_C g\,dy$

AND $\oint_C f\,dx + g\,dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA$