Integrals for mass calculations

Objective:

1. Review of section 8.3
2. Density and mass
3. Moments and center of mass
4. Moments of inertia

1. Review of Section 8.3 from calc II.

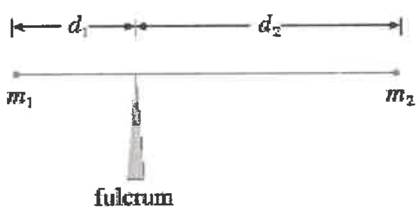
Mass: A body of matter with no definite shape. We use m to denote it.

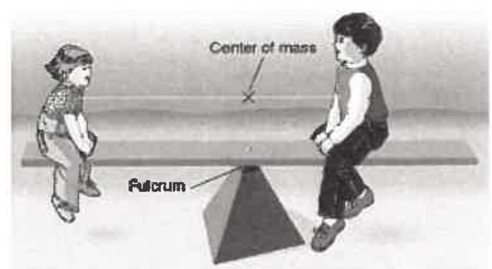
Density: The degree of compactness of a substance. We use ρ to denote it.

Center of Mass: The point on which a thin plate balances horizontally.

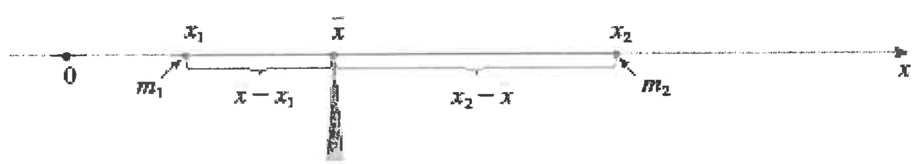
The Law of the Lever: Suppose masses m_1 and m_2 are attached to a rod of negligible mass. The rod will balance if:

$$m_1 d_1 = m_2 d_2$$





Now suppose that the rod lies along the x-axis, m_1 is at x_1, m_2 is at x_2, and center of mass is at $\bar{x}$.



Then:

$$m_1(\bar{x}-x_1) = m_2(x_2-\bar{x})$$

$$m_1\bar{x} + m_2\bar{x} = m_1x_1 + m_2x_2$$

$$\bar{x} = \frac{m_1x_1 + m_2x_2}{m_1+m_2}$$

Moments: A turning effect produced by a force acting at a distance on an object. The numbers m_1x_1 and m_2x_2 are called the moments of the particles with masses m_1 and m_2 with respect to the origin.

In general, if we have a system of n particles with masses $m_1, m_2, \dots, m_n$ located at the points $x_1, x_2, \dots, x_n$ on the x-axis, the center of mass of the system is located at:

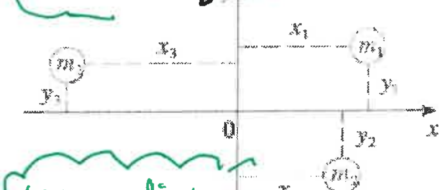
$$\bar{x} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i} = \frac{M}{m}$$

Where m is the total mass of the system and M is the moment of the system about the origin.

System in two dimensions, made out of a countable number of particles

The moment of the system about the y-axis measures the tendency of the system to rotate about the y-axis: $M_y = \sum_{i=1}^{n} m_i x_i$

x_i = distance from the y-axis



The moment of the system about the x-axis measures the tendency of the system to rotate about the x-axis: $M_x = \sum_{i=1}^{n} m_i y_i$

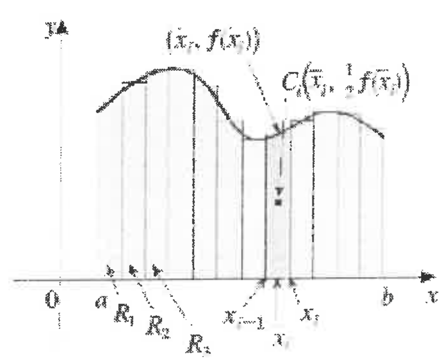
y_i = distance from the x-axis

The coordinates of the center of mass are given by $(\bar{x}, \bar{y}) = \left(\frac{M_y}{m}, \frac{M_x}{m}\right)$

System in two dimensions, made out of non-countable number of particles

Consider a flat plate with uniform density ρ. The plate is made out of infinitely many particles. To find its center of mass (or centroid) we need to divide the plate into very small rectangles (particle size!), find the center of mass (or centroid) of each rectangle, and add them all together. The center of mass is called the centroid. Note: The centroid of a rectangular plate with uniform density is its center (where the diagonals meet).

Let's find the moments and center of mass of the region under the curve $f(x)$ from a to b. The i[th] subinterval has:



Midpoint: $\bar{x}_i = \frac{(x_{i-1} + x_i)}{2}$ Center (centroid): $C_i\left(\bar{x}_i, \frac{1}{2} f(\bar{x}_i)\right)$

Area: $f(\bar{x}_i)\Delta x$ Mass: $\rho f(\bar{x}_i)\Delta x$

The moment of the region about y-axis:

$$M_y = \lim_{n \to \infty} \sum_{i=1}^{n} \rho \bar{x}_i f(\bar{x}_i)\Delta x = \rho \int_a^b x f(x)\, dx$$

The Moment of the region about x-axis:

$$M_x = \lim_{n\to\infty} \sum_{i=1}^{n} \rho \cdot \tfrac{1}{2}[f(\bar{x}_i)]^2 \Delta x = \rho \int_a^b \tfrac{1}{2}[f(x)]^2 dx$$

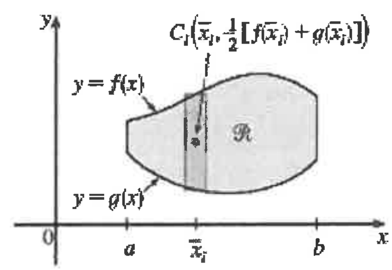
Since the centroid is $(\bar{x}, \bar{y}) = \left(\frac{M_y}{m}, \frac{M_x}{m}\right)$ and mass of the plate is the product of its density and its area, $m = \rho A = \rho \int_a^b f(x)\,dx$, the coordinates of the centroid of the region or center of the mass of the plate can be expressed as:

$$\bar{x} = \frac{1}{A}\int_a^b x f(x)\,dx \qquad \bar{y} = \frac{1}{A}\int_a^b \tfrac{1}{2}[f(x)]^2\,dx$$

If the region lies between two curves, the coordinates of the centroid of the region or center of the mass of the plate: are;

$$\bar{x} = \frac{1}{A}\int_a^b x[f(x) - g(x)]\,dx$$

$$\bar{y} = \frac{1}{A}\int_a^b \tfrac{1}{2}\{[f(x)]^2 - [g(x)]^2\}\,dx$$



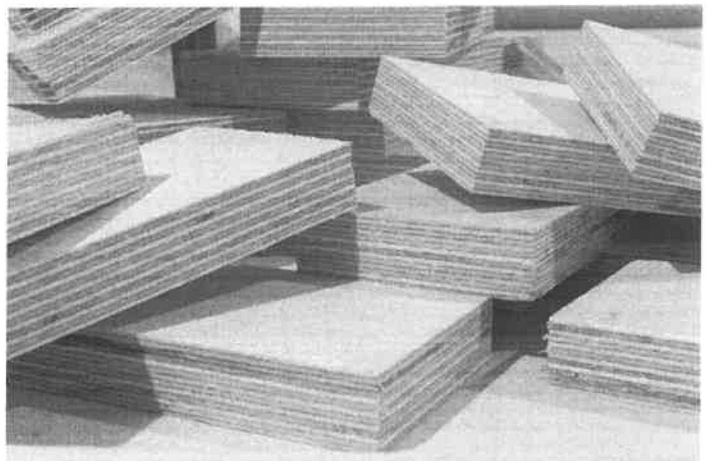
There is a surprising connection between centroids and volumes of revolution:

> **Theorem of Pappus** Let $\mathcal{R}$ be a plane region that lies entirely on one side of a line l in the plane. If $\mathcal{R}$ is rotated about l, then the volume of the resulting solid is the product of the area A of $\mathcal{R}$ and the distance d traveled by the centroid of $\mathcal{R}$.

2. *Density and Mass in non-uniform objects*

To begin, let's consider an idealized flat object that is sufficiently thin that it can be viewed as two-dimensional. Such an object is called ***Lamina***. A lamina is called **homogeneous** if its composition and structure are uniform throughout.

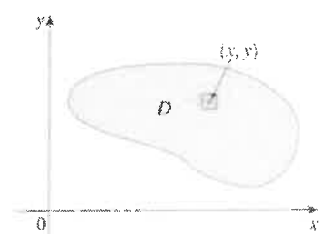
lamina is like plywood



In calculus II we saw that the density of a homogeneous lamina is a constant and defined to be its mass per unit area:

$$\rho = \frac{m}{A} \qquad \text{which gives} \qquad m = \rho A$$

Now we will look at lamina with a variable density (that means the composition (density) may vary from point to point). To define a function for this variable density, consider the lamina to be on the xy-plane and the point (x, y) to be on the lamina. Construct a small rectangle centered at (x, y), with mass Δm and area ΔA. If the ratio $\frac{\Delta m}{\Delta A}$ approaches a limiting value as the dimensions (and hence the area) of the rectangle approaches zero, then this limit is considered to be the density of lamina at (x, y).



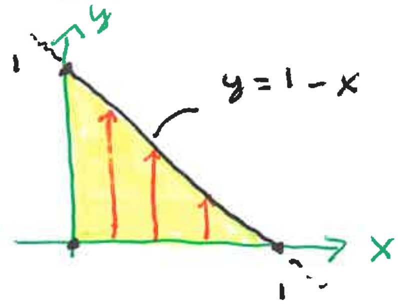
$$\rho(x, y) = \lim_{\Delta A \to 0} \frac{\Delta m}{\Delta A}$$

From this relationship we obtain the approximation $\Delta m \approx \rho(x, y)\Delta A$

As the dimensions of the rectangle tends to zero, the error to this approximation tends to zero as well.

<u>Ex1</u>: A triangular lamina with vertices $(0,0),(0,1)$ and $(1,0)$ has density function $\rho(x,y)=xy$. Find its total mass.

<u>Step 1: Draw a pic</u>



<u>Step 2:</u> set-up & evaluate an integral.

$$\text{mass} = \iint_{\triangle} \rho(x,y)\,dA$$

$$= \int_0^1 \int_{y=0}^{y=1-x} xy \; dy\,dx$$

$$= \int_0^1 \left[\frac{1}{2}xy^2\right]_{y=0}^{y=1-x} dx$$

$$= \int_0^1 \frac{1}{2}\underbrace{x(1-x)^2}_{x^3-2x^2+x}\,dx$$

$$= \frac{1}{2}\left[\frac{1}{4}x^4 - \frac{2}{3}x^3 + \frac{1}{2}x^2\right]_0^1$$

$$= \frac{1}{2}\left(\frac{1}{4} - \frac{2}{3} + \frac{1}{2}\right)$$

$$= \frac{1}{24}$$

Physicists also consider other types of density that can be treated in the same manner. For example, if an electric charge is distributed over a region D and the charge density (in units of charge per unit area) is given by $\sigma(x, y)$ at a point (x, y) in D, then the total charge Q is given by

$$Q = \iint_D \sigma(x, y)\, dA$$

3. *Moments and Center of Mass*

The moment of a particle about an axis can be found as the product of its mass and its directed distance from the axis. To approximate the total moment of a lamina we divide the lamina into small rectangles and add their moments together.

About x-axis: $M_x \approx \sum_{i=1}^{m}\sum_{j=1}^{n} y_{ij}^*\rho\left(x_{ij}^*, y_{ij}^*\right)\Delta A$

About y-axis: $M_y \approx \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij}^*\rho\left(x_{ij}^*, y_{ij}^*\right)\Delta A$

As the number of rectangles approaches ∞ we can find the total moment of the lamina.

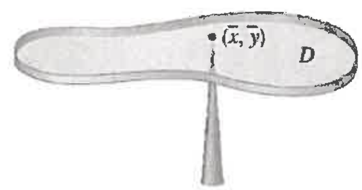
Moment **About the x-axis:** $\circ\circ\circ$ {tendency to rotate about the x-axis}

$$\boxed{M_x = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} y_{ij}^* \, \rho(x_{ij}^*, y_{ij}^*)\, \Delta A = \iint_D y\,\rho(x,y)\,dA}$$

Moment **About the y-axis:** $\circ\circ\circ$ {tendency to rotate about the y-axis.}

$$\boxed{M_y = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij}^* \, \rho(x_{ij}^*, y_{ij}^*)\, \Delta A = \iint_D x\,\rho(x,y)\,dA}$$

A lamina balances horizontally when supported at its center of mass



$$\left(\bar{x}, \bar{y}\right) = \left(\frac{M_y}{m}, \frac{M_x}{m}\right).$$

Hence:

$$\boxed{\bar{x} = \frac{M_y}{m} = \frac{1}{m}\iint_D x\,\rho(x,y)\,dA \qquad \bar{y} = \frac{M_x}{m} = \frac{1}{m}\iint_D y\,\rho(x,y)\,dA}$$

Ex 1 rev: Find the center of mass of a triangular lamina in example 1.

$$M_y = \int_0^1 \int_0^{1-x} x \cdot xy \, dy\, dx$$

$$= \int_0^1 \left[x^2 \cdot \frac{1}{2}(1-x)^2\right] dx$$

$$= \int_0^1 \frac{1}{2}(x^4 - 2x^3 + x^2)\,dx$$

$$= \frac{1}{2}\left[\frac{1}{5}x^5 - \frac{2}{4}x^4 + \frac{1}{3}x^3\right]_0^1$$

$$= \frac{1}{2}\left(\frac{1}{5} - \frac{1}{2} + \frac{1}{3}\right)$$

$$\Rightarrow \quad M_y = \frac{1}{60}$$

$$\text{And} \quad \bar{X} = \frac{M_y}{m} = \frac{\frac{1}{60}}{\frac{1}{24}} = \frac{2}{5}$$

$$M_x = \int_0^1 \int_0^{1-x} y \cdot xy \, dy \, dx$$

$$= \int_0^1 \left[x \cdot \frac{1}{3} y^3 \right]_0^{1-x} dx$$

$$= \frac{1}{3} \int_0^1 x \, \underset{\substack{\uparrow \\ 1 - 3x + 3x^2 - x^3}}{(1-x)^3} \, dx$$

$$= \frac{1}{3} \int_0^1 (x - 3x^2 + 3x^3 - x^4) \, dx$$

$$= \frac{1}{3} \left[\frac{1}{2} x^2 - x^3 + \frac{3}{4} x^4 - \frac{1}{5} x^5 \right]_0^1$$

$$= \frac{1}{3} \left(\frac{1}{2} - 1 + \frac{3}{4} - \frac{1}{5} \right)$$

$$\Rightarrow M_x = \frac{1}{60}$$

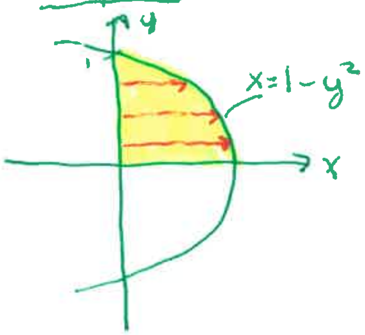
And $\bar{y} = \frac{2}{5}$ ... but we already knew this thru. symmetry

Conclusion: The centroid is @ $\left(\frac{2}{5}, \frac{2}{5} \right)$

Ex2: Consider a lamina that occupies the region bounded by the parabola $x = 1 - y^2$ and the coordinate ~~axis~~ axes in the first quadrant with density function $\rho(x, y) = y$.

a) Find the mass of the lamina.

Step 1: draw a pic



Step 2 set-up and integrate

$$m = \int_0^1 \int_{x=0}^{x=1-y^2} y \, dx\, dy$$

$$= \int_0^1 \left[xy \right]_{x=0}^{x=1-y^2} dy$$

$$= \int_0^1 y - y^3 \, dy$$

$$= \left[\tfrac{1}{2} y^2 - \tfrac{1}{4} y^4 \right]_0^1$$

$$m = \frac{1}{4}$$

b) Find the center of mass.

$$M_y = \int_0^1 \int_0^{1-y^2} x \, y \, dx\, dy$$

$$= \int_0^1 \left[\tfrac{1}{2} x^2 y \right]_0^{1-y^2} dy$$

$$= \frac{1}{2} \int_0^1 (1-y^2)^2 y \, dy$$

$$= \frac{1}{2} \int_0^1 y - 2y^3 + y^5 \, dy$$

$$= \frac{1}{2} \left[\tfrac{1}{2} y^2 - \tfrac{2}{4} y^4 + \tfrac{1}{6} y^6 \right]_0^1$$

$$= \frac{1}{2} \left(\frac{1}{2} - \frac{1}{2} + \frac{1}{6} \right)$$

$$= \frac{1}{12}$$

$$\Rightarrow \bar{x} = \frac{1/12}{1/4} = \frac{1}{3}$$

$$M_x = \int_0^1 \int_0^{1-y^2} y \cdot y \, dx\, dy$$

$$= \int_0^1 \left[x y^2 \right]_{x=0}^{x=1-y^2} dy$$

$$= \int_0^1 y^2 - y^4 \, dy$$

$$= \left[\tfrac{1}{3} y^3 - \tfrac{1}{5} y^5 \right]_0^1$$

$$= \frac{1}{3} - \frac{1}{5}$$

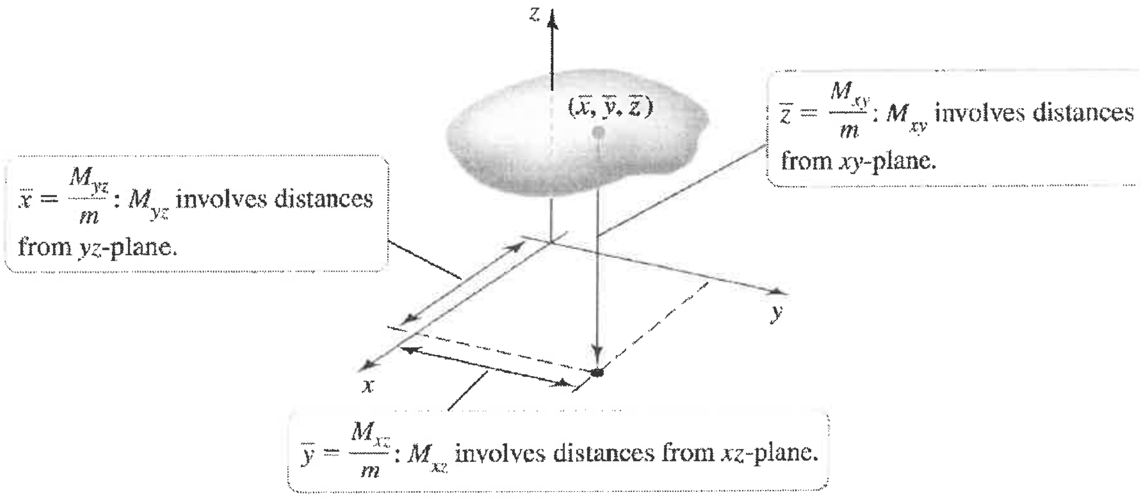
$$= \frac{2}{15}$$

$$\Rightarrow \bar{y} = \frac{2/15}{1/4} = \frac{8}{15}$$

Conclusion: the centroid is @ $\left(\frac{1}{3}, \frac{8}{15} \right)$

4. *Moments and Center of Mass in 3D*

We can use the same reasoning to find the mass, moments, and centroid in three dimensions.



This leads to the following formulas (analogs to our work in 2D)

DEFINITION **Center of Mass in Three Dimensions**

Let ρ be an integrable density function on a closed bounded region D in $\mathbb{R}^3$. The coordinates of the center of mass of the region are

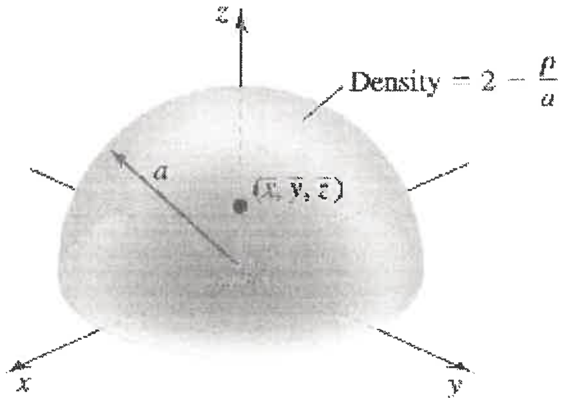
$$\bar{x} = \frac{M_{yz}}{m} = \frac{1}{m}\iiint_D x\,\rho(x, y, z)\,dV,$$

$$\bar{y} = \frac{M_{xz}}{m} = \frac{1}{m}\iiint_D y\,\rho(x, y, z)\,dV,$$

$$\bar{z} = \frac{M_{xy}}{m} = \frac{1}{m}\iiint_D z\,\rho(x, y, z)\,dV,$$

where $m = \iiint_D \rho(x, y, z)\,dV$ is the mass, and M_{yz}, M_{xz}, and M_{xy} are the moments with respect to the coordinate planes.

<u>Ex3</u>: Find the center of mass of the interior of the hemisphere D of radius a with its base on the xy-plane. The density is $f(\rho,\phi,\theta) = 2 - \frac{\rho}{a}$ (heavy near the center and light near the outer surface);



$$m = \int_0^{\frac{\pi}{2}}\int_0^{2\pi}\int_0^a \left(2 - \frac{\rho}{a}\right)\rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

$$= \underbrace{\int_0^{\frac{\pi}{2}} \sin\phi \, d\phi}_{1} \; \underbrace{\int_0^{2\pi} 1 \, d\theta}_{2\pi} \; \underbrace{\int_0^a \left(2\rho^2 - \frac{\rho^3}{a}\right) d\rho}_{\left[\frac{2}{3}\rho^3 - \frac{\rho^4}{4a}\right]_0^a}$$

random fact I learned

$$= 2\pi\left(\frac{2}{3}a^3 - \frac{1}{4}a^3\right)$$

$$= \frac{5}{6}\pi$$