HW: 72 min + 3Q's

Triple Integrals

Objective:

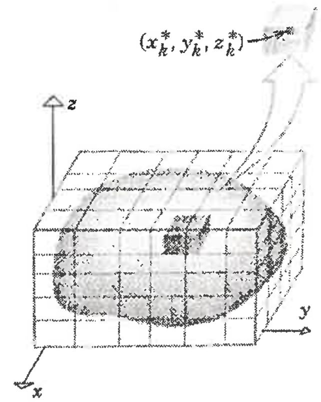
1. Definition of triple integrals
2. Triple integrals over a general solid
3. Applications of triple integrals

I. Definition of Triple Integrals

check out manipulate 16.38

Just as we defined single integrals for functions of one variable over an axis and double integrals for functions of two variables over a closed region on xy-plane, we can define triple integrals for functions of three variables over a closed three-dimensional solid.

To define triple-integrals, we'll first divide the solid into small boxes with sides parallel to the coordinate planes. Each of these small boxes have volume: $\Delta V = \Delta x \Delta y \Delta z$. As we did for two and three variable functions we multiply ΔV by the value of the function, for a sample point, in each box. Then adding them all together we form the triple Riemann sum:



$$\sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)\, \Delta V$$

Making the size of the boxes smaller and smaller (by allowing the number of boxes to grow infinitely larger) we will have:

> **Definition** The **triple integral** of f over the box B is
>
> $$\iiint_B f(x, y, z)\, dV = \lim_{l, m, n \to \infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)\, \Delta V$$
>
> if this limit exists.

This is solved analogously to double integrals where $dV = dx\,dy\,dz$ parallels the formula $dA = dx\,dy$. If the solid B is a box, the integrals are much easier:

> **Fubini's Theorem for Triple Integrals** If f is continuous on the rectangular box $B = [a, b] \times [c, d] \times [r, s]$, then
>
> $$\iiint_B f(x, y, z)\, dV = \int_r^s \int_c^d \int_a^b f(x, y, z)\, dx\, dy\, dz$$

Otherwise we have to be very careful in determining the limits of integration. In this course we will only consider continues functions over simple smooth solids.

Ex1: Evaluate $\iiint\limits_B 2-z \, dv$ over the rectangular box $0 \le x \le 3$ $\quad 0 \le y \le 2$ $\quad 0 \le z \le 1$

$$\iiint\limits_B 2-z \, dV = \int_0^3 \int_0^2 \underbrace{\int_0^1 2-z \, dz}_{\left[2z - \frac{z^2}{2}\right]_0^1} \, dy \, dx$$

$$= \int_0^3 \int_0^2 \frac{3}{2} \, dy \, dx$$

oooO! No x or y in the limits or integrand

$$= 6 \cdot \frac{3}{2}$$

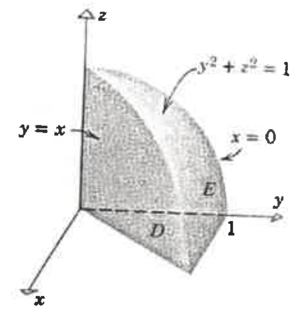
$$= 9$$

check out manipulate 16.41

Determining the limits of integration

- Draw a picture of the 3D region over which you are integrating.
- Inner limits:
  - On the 3D model, sketch an arrow parallel to the axis of the inner variable. The arrow enters the model at the lower limit and exits at the upper limit.
- Draw a second 2D picture. This sketch is of the projection of the 3D object onto the plane formed by the outer two variables.
- Middle limits:
  - Sketch three arrows parallel to the axis of the middle variable on the 2D picture. The arrow enters the model at the lower limit and exits at the upper limit.
- Outer limits:
  - On the 2D picture, you should have a left/bottom – middle – right/top arrow.
    - The lower limit would come from the leftmost/lowest possible such arrow.
    - The upper limit would come from the rightmost/highest possible such arrow.

Ex2: Let E be the wedge in the first octant cut from the cylindrical solid $y^2+z^2\le 1$ by the planes $y=x$ and $x=0$. Evaluate $\iiint_E z\,dv$.

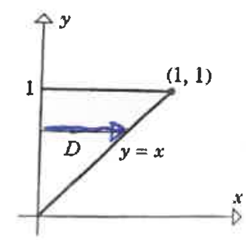


$$I=\int_0^1\int_0^y\int_0^{\sqrt{1-y^2}} z\,dz\,dx\,dy$$

$$=\int_0^1\int_0^y\left[\frac{1}{2}z^2\right]_0^{\sqrt{1-y^2}}dx\,dy$$

$$=\int_0^1\int_0^y\frac{1}{2}(1-y^2)\,dx\,dy$$

No x's in integrand



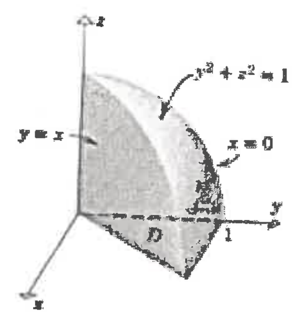
$$=\int_0^1\frac{1}{2}(1-y^2)\,y\,dy$$

$$=\frac{1}{2}\int_0^1 y-y^3\,dy$$

$$=\frac{1}{2}\left[\frac{1}{2}y^2-\frac{1}{4}y^4\right]_0^1$$

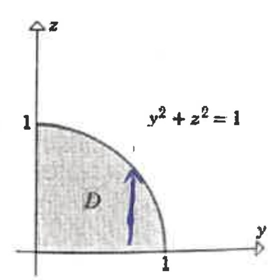
thus $I=\frac{1}{2}\left(\frac{1}{2}-\frac{1}{4}\right)=\frac{1}{8}$

Ex3: Go back to example 2 but this time evaluate $\iiint_E z\,dv$ with respect to x first.



$$I=\int_0^1\int_0^{\sqrt{1-y^2}}\underbrace{\int_0^y z\,dx}_{[zx]_0^y}\,dz\,dy$$

$$=\underbrace{\int_0^1\int_0^{\sqrt{1-y^2}} zy\,dz}_{\left[\frac{1}{2}z^2y\right]_0^{\sqrt{1-y^2}}}\,dy$$



$$=\int_0^1\frac{1}{2}(1-y^2)y\,dy \quad \text{(same as above).}$$

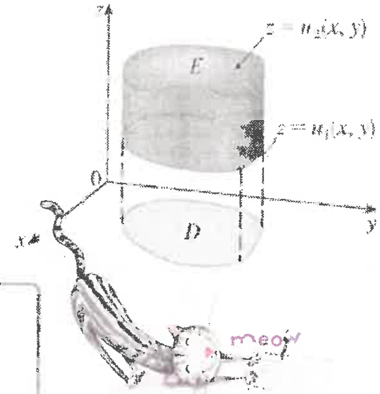
$$=\frac{1}{8}$$

2. ***Triple Integrals over a General Solid*** (for those who like memorization)

<u>Type1</u>: When the solid E is bounded between two continuous functions $z = u_1(x,y)$ and $z = u_2(x,y)$ we describe E as:

$$E = \{(x,y,z) \mid (x,y) \in D \text{ and } u_1(x,y) \le z \le u_2(x,y)\}$$

where D is the projection of E onto *xy*-plane.



$$\iiint_E f(x,y,z)\,dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz \right] dA$$

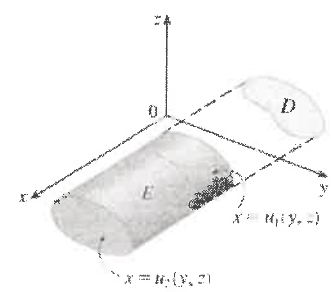
check out manipulate 16.40

The first (innermost) integration is with respect to z, after that a function of x and y remains. This function then gets integrated over region D in *xy*-plane which can be evaluated as we learned in the calculus III as a type I or II double integral.

<u>Type2</u>: When the solid E is bounded between two continuous functions $x = u_1(y,z)$ and $x = u_2(y,z)$ we describe E as:

$$E = \{(x,y,z) \mid (y,z) \in D \text{ and } u_1(y,z) \le x \le u_2(y,z)\}$$

where D is the projection of E onto *yz*-plane.

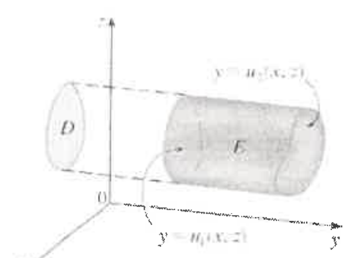


$$\iiint_E f(x,y,z)\,dV = \iint_D \left[\int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)\,dx \right] dA$$

<u>Type3</u>: When the solid E is bounded between two continuous functions $y = u_1(x,z)$ and $y = u_2(x,z)$ we describe E as:

$$E = \{(x,y,z) \mid (x,z) \in D \text{ and } u_1(x,z) \le y \le u_2(x,z)\}$$

where D is the projection of E onto *xz*-plane.



$$\iiint_E f(x,y,z)\,dV = \iint_D \left[\int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)\,dy \right] dA$$

Sometimes you have a choice to choose between type 1, 2 or 3 and may find one type easier. This is a case where practice is superior to memorization.

3. *Applications of Triple Integrals*

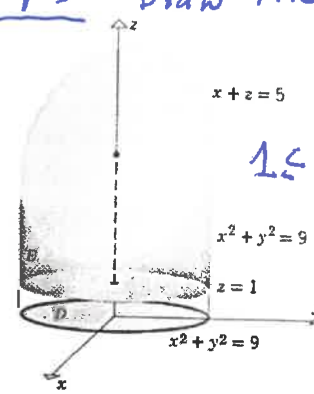
Recall that if $f(x) \geqslant 0$, then the single integral $\int_a^b f(x)\,dx$ represents the area under the curve $y = f(x)$ from a to b, and if $f(x, y) \geqslant 0$, then the double integral $\iint_D f(x, y)\,dA$ represents the volume under the surface $z = f(x, y)$ and above D. The corresponding interpretation of a triple integral $\iiint_E f(x, y, z)\,dV$, where $f(x, y, z) \geqslant 0$, is not very useful because it would be the "hypervolume" of a four-dimensional object and, of course, that is very difficult to visualize. (Remember that E is just the *domain* of the function f; the graph of f lies in four-dimensional space.) Nonetheless, the triple integral $\iiint_E f(x, y, z)\,dV$ can be interpreted in different ways in different physical situations, depending on the physical interpretations of x, y, z, and $f(x, y, z)$.

Let's begin with the special case where $f(x, y, z) = 1$ for all points in E. Then the triple integral does represent the volume of E:

$$\boxed{V(E) = \iiint_E dV}$$

Ex4: Use a triple integral to find the volume of the solid enclosed between the cylinder $x^2 + y^2 = 9$ and the planes $z = 1$ and $x + z = 5$.

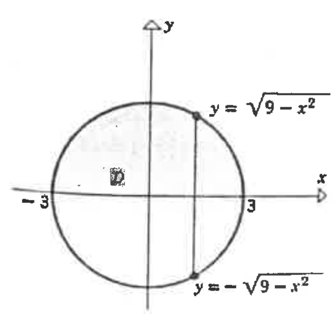
Step 1: Draw the pic



$1 \leq z \leq 5 - x$

$$V = \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\underbrace{\int_{1}^{5-x} 1\,dz}_{\left[z\right]_1^{5-x}}\,dy\,dx$$

$$= \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} (4 - x)\,dy\,dx$$



switch to polar

$$= \int_{0}^{2\pi}\int_{0}^{3} (4 - r\cos\theta)\,r\,dr\,d\theta$$

$$= \int_{0}^{2\pi}\underbrace{\int_{0}^{3} (4r - r^2\cos\theta)\,dr}_{\left[2r^2 - \frac{r^3}{3}\cos\theta\right]_0^3}\,d\theta$$

$-\sqrt{9-x^2} \leq y \leq \sqrt{9-x^2}$

$-3 \leq x \leq 3$

$$= \int_0^{2\pi} (18 - 9\cos\theta)\, d\theta$$

$$= 36\pi - 9 \underbrace{\int_0^{2\pi} \cos\theta \, d\theta}_{0}$$

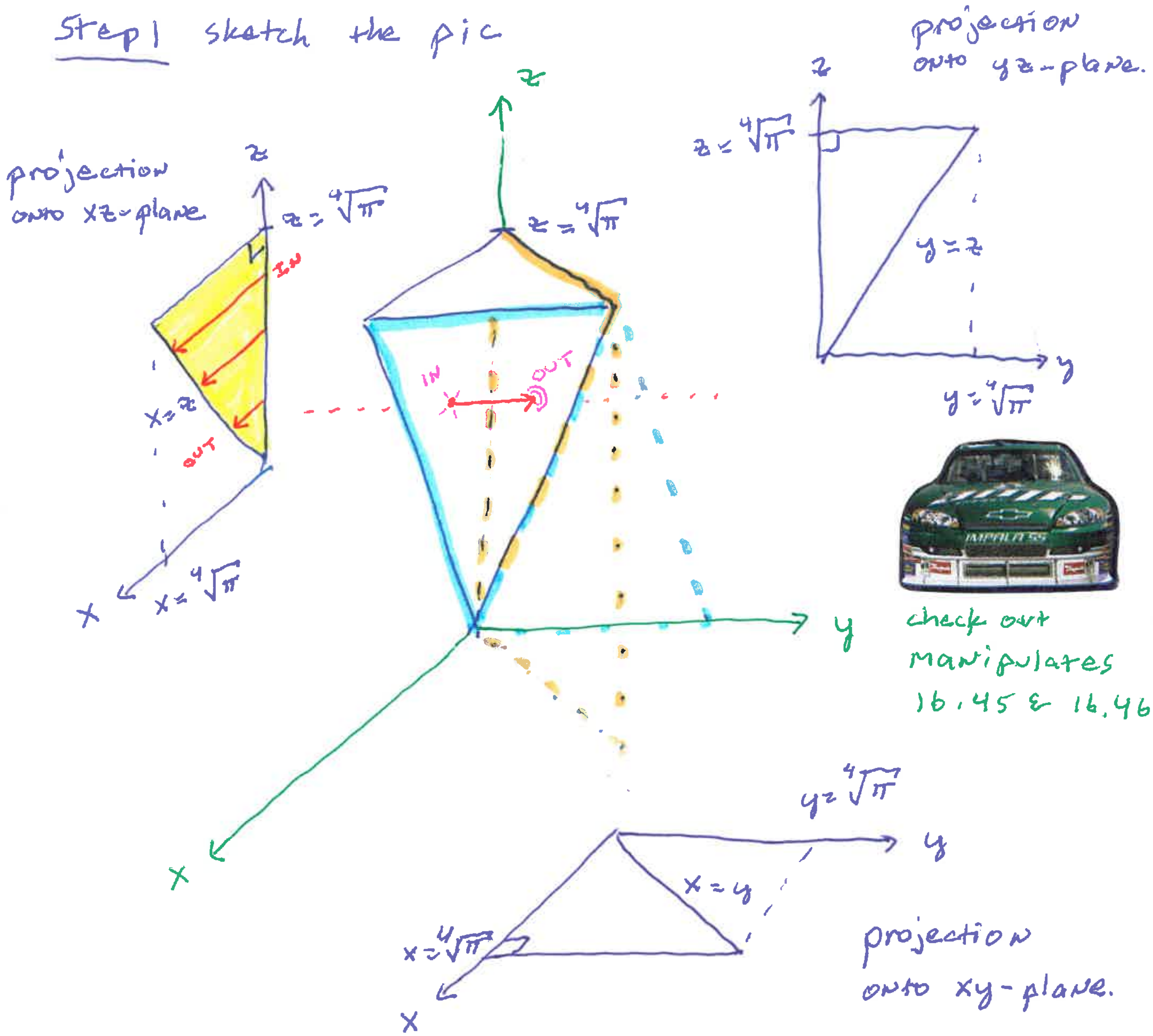
$$= 36\pi.$$

Ex4: Consider the integral $I = \underbrace{\int_0^{\sqrt[4]{\pi}} \int_0^{z} \int_y^{z} 12y^2 z^3 \sin\left(x^4\right) dx\, dy\, dz}$

we need the antiderivative of $\sin(x^4)$.
The only way we know for finding this is using a Maclaurin series from calc III...

so let's change the order of integration.

Step 1 sketch the pic



check out manipulates 16.45 & 16.46

Step 2: Limits

$$\underset{\substack{\uparrow\\ y=0\\ \text{IN}}}{0} \le y \le \underset{\substack{\nwarrow\\ y=x\\ \text{OUT}}}{x}$$

From 3D pic

$$\underset{\substack{\uparrow\\ x=0\\ \text{IN}}}{0} \le x \le \underset{\substack{\nwarrow\\ x=z\\ \text{OUT}}}{z}$$

$$\underset{\substack{\uparrow\\ z=0\\ \text{lowest}\\ \text{possible}\\ \text{arrow}}}{0} \le z \le \underset{\substack{\nwarrow\\ z=\sqrt[4]{\pi}\\ \text{highest}\\ \text{possible}\\ \text{arrow}}}{\sqrt[4]{\pi}}$$

From 2D pic

Step 3: Setup and integrate.

$$I = \int_0^{\sqrt[4]{\pi}}\int_0^{z}\underbrace{\int_0^{x} 12y^2z^3\sin(x^4)\,dy}_{\left[4y^3z^3\sin(x^4)\right]_{y=0}^{y=x}}\,dx\,dz$$

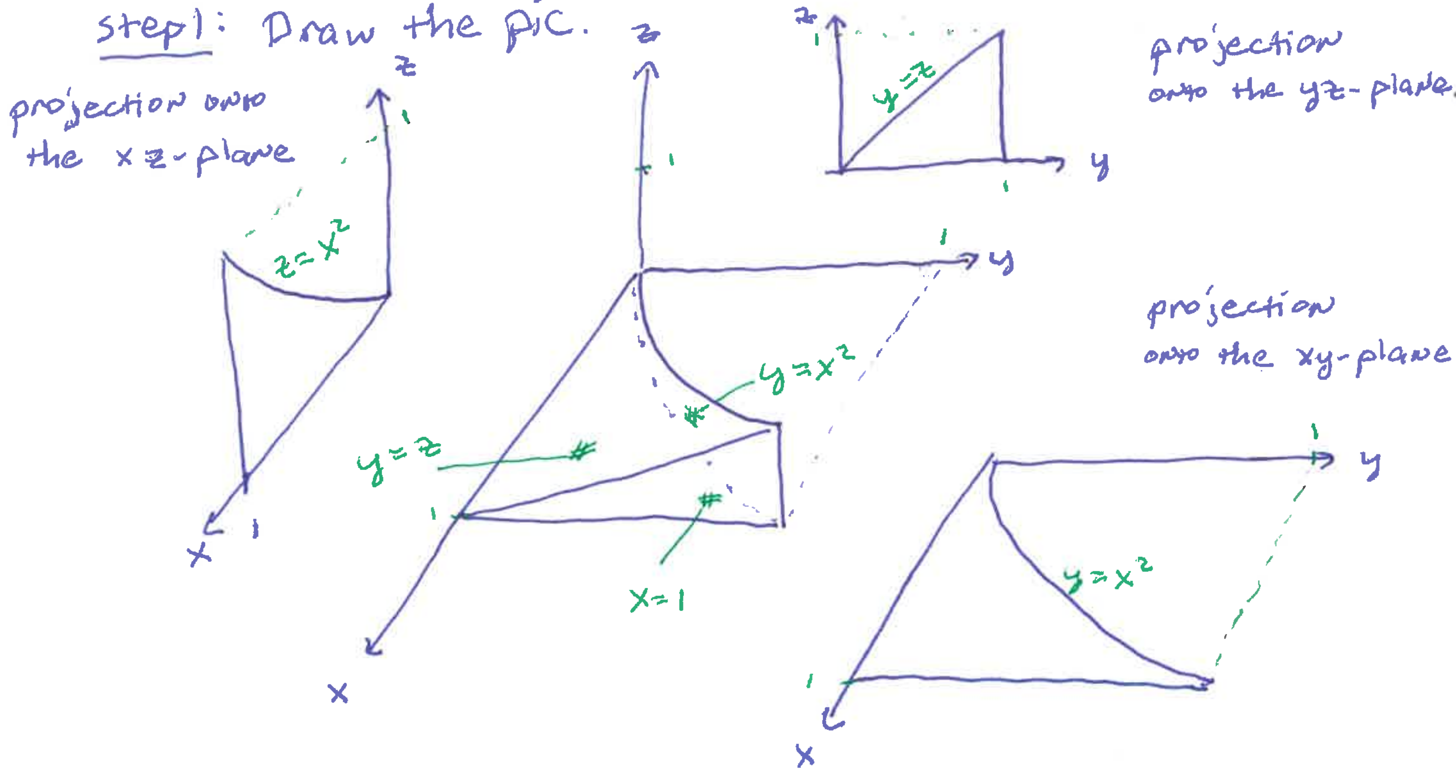
$$= \int_0^{\sqrt[4]{\pi}}\underbrace{\int_0^{z} 4x^3z^3\sin(x^4)\,dx}_{\left[-z^3\cos(x^4)\right]_{x=0}^{x=z}}\,dz$$

$$= \int_0^{\sqrt[4]{\pi}}\left(z^3 - z^3\cos(z^4)\right)dz$$

$$= \left[\frac{z^4}{4} - \frac{1}{4}\sin(z^4)\right]_{z=0}^{z=\sqrt[4]{\pi}} \qquad \frac{\pi}{4}$$

<u>Ex5</u>: Write five other iterated integrals that are equal to $\int_0^1 \int_0^{x^2} \int_0^y f(x,y,z)\,dz\,dy\,dx$

Step 1: Draw the pic.



$$\text{(A.)} \quad \int_0^1 \int_{\sqrt{y}}^1 \int_0^y f \; dz\,dx\,dy$$

$$\text{(B.)} \quad \int_0^1 \int_{\sqrt{z}}^1 \int_z^{x^2} f \; dy\,dx\,dz$$

$$\text{(C.)} \quad \int_0^1 \int_0^{x^2} \int_z^{x^2} f \; dy\,dz\,dx$$

$$\text{(D.)} \quad \int_0^1 \int_0^y \int_{\sqrt{y}}^1 f \; dx\,dz\,dy$$

$$\text{(E.)} \quad \int_0^1 \int_z^1 \int_{\sqrt{y}}^1 f \; dx\,dy\,dz$$