

Double Integrals in Polar Coordinates

Objective:

- 1. Double Integrals in Polar Coordinates

Important Formulas we will need:

RELATIONSHIP BETWEEN POLAR AND RECTANGULAR COORDINATES

1. To change from polar to rectangular coordinates, use the formulas

$$x = r \cos \theta \quad \text{and} \quad y = r \sin \theta$$

2. To change from rectangular to polar coordinates, use the formulas

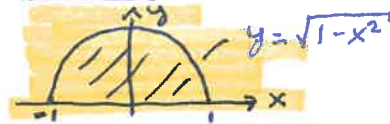
$$r^2 = x^2 + y^2 \quad \text{and} \quad \tan \theta = \frac{y}{x} \quad (x \neq 0)$$

1. Double Integrals in Polar Coordinates

In this section we will study double integrals in which the integrand and the region of integration are expressed in polar coordinates. Such integrals are important for two reasons: First, they arise naturally in many applications, and second, many double integrals in rectangular coordinates are more easily evaluated if they are converted to polar coordinates.

Ex 1: Evaluate $\int_{-1}^1 \int_0^{\sqrt{1-x^2}} (x^2 + y^2)^{3/2} dy dx$

step 1: sketch the region.



The region is a half disk... nice in polar coords.

Step 2: Antidifferentiate

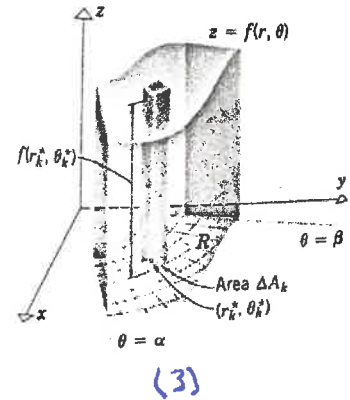
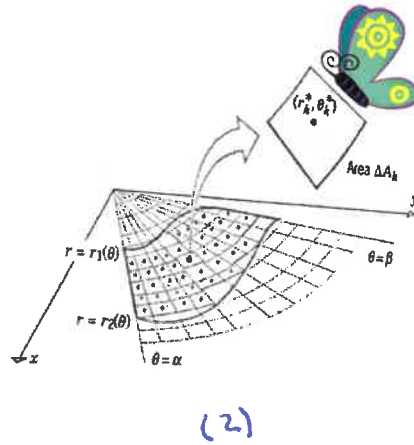
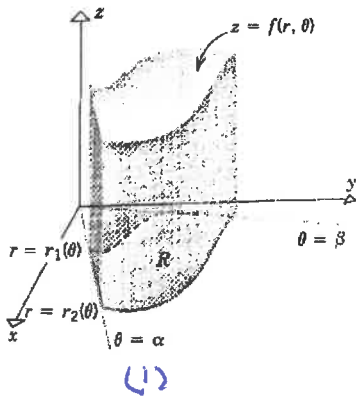
This would require trig sub @ the 1st step!

who knows what would come next.

However, we notice that $x^2 + y^2 = r^2$... nice in polar coords.

To find the volume of a solid bounded by the region $R = \{(r, \theta) \mid r_1(\theta) \leq r \leq r_2(\theta), \alpha \leq \theta \leq \beta\}$ in the xy -plane and a surface $z = f(r, \theta)$, we follow the same method as previous sections. (1) Dividing R into n small polar rectangles (circular arcs and rays) of area ΔA_k , (2) picking a sample point in each rectangle, (3) Find the volume of each cylinder by $f(r_k^*, \theta_k^*) \Delta A_k$, (4) adding the volumes, (5) and finally allowing $n \rightarrow \infty$.

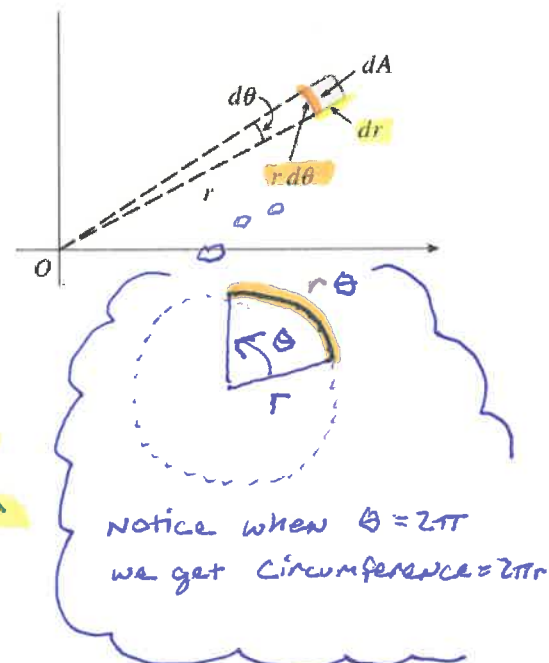
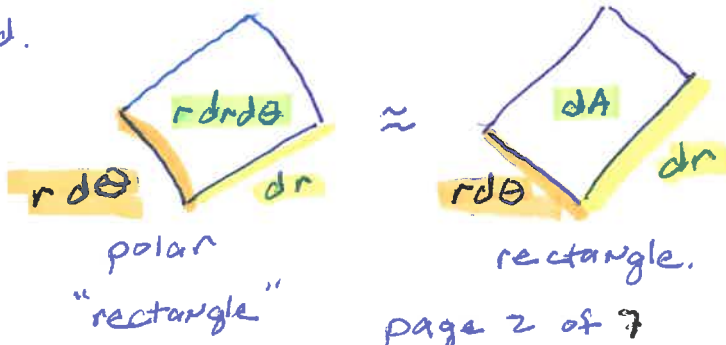
see manipulate 16.29



So the volume of the solid can be expressed as: $V = \iint_R f(r, \theta) dA$. But how do we calculate dA (the area of the polar rectangles)?

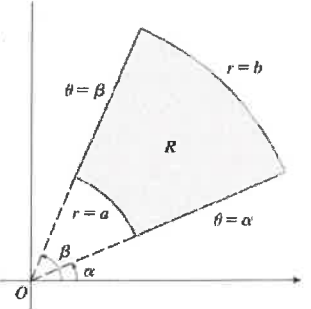
We will calculate dA intuitively.

Consider the small polar rectangle pictured.



Change to Polar Coordinates in a Double Integral If f is continuous on a polar rectangle R given by $0 \leq a \leq r \leq b$, $\alpha \leq \theta \leq \beta$, where $0 \leq \beta - \alpha \leq 2\pi$, then

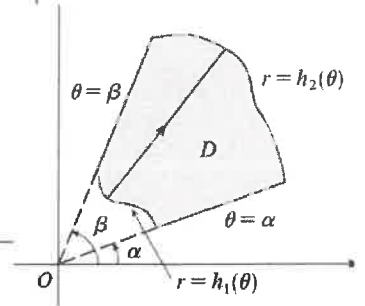
$$\iint_R f(x, y) dA = \int_{\alpha}^{\beta} \int_a^b f(r \cos \theta, r \sin \theta) r dr d\theta$$



If f is continuous on a polar region of the form

$$D = \{(r, \theta) \mid \alpha \leq \theta \leq \beta, h_1(\theta) \leq r \leq h_2(\theta)\}$$

then

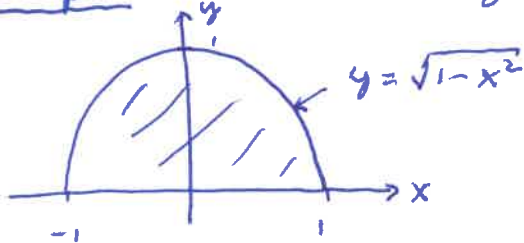
$$\iint_D f(x, y) dA = \int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} f(r \cos \theta, r \sin \theta) r dr d\theta$$


See manipulate 16,34

Now let's go back to example 1 and use polar coordinates to evaluate it!

$$I = \int_{-1}^1 \int_0^{\sqrt{1-x^2}} (x^2 + y^2)^{3/2} dy dx$$

Step 1: sketch the region



Step 2: find the limits of integration in polar coordinates

$$0 \leq r \leq 1$$

$$0 \leq \theta \leq \pi$$

Step 3: set up and evaluate the integral.

$$I = \int_0^{\pi} \int_0^1 (r^2)^{3/2} r dr d\theta$$

$$= \int_0^{\pi} \int_0^1 r^4 dr d\theta$$

$$= \int_0^{\pi} \left[\frac{1}{5} r^5 \right]_0^1 d\theta$$

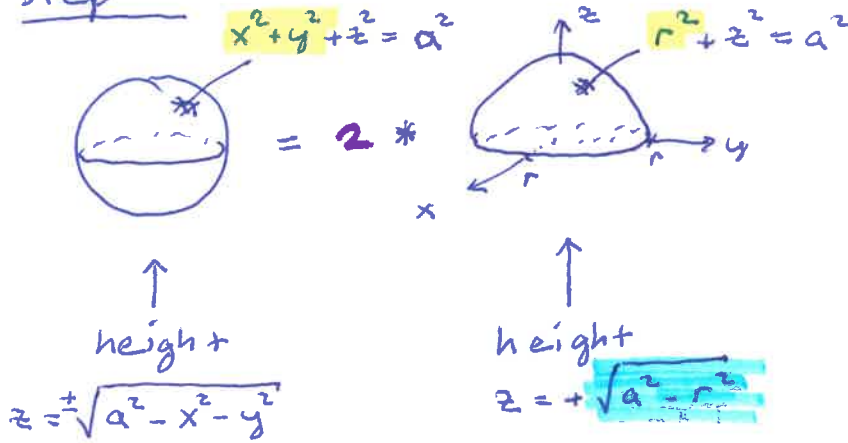
$$= \int_0^{\pi} \frac{1}{5} d\theta$$

$$= \frac{1}{5} \pi.$$

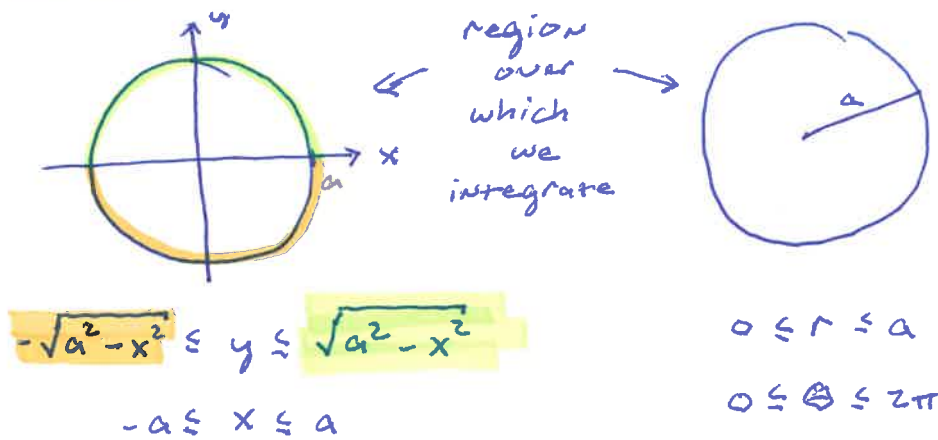
Ex2: Use a double polar integral to formulate the volume of a sphere of radius a .

$V = \frac{4}{3} \pi r^3$
volume of a sphere

step 1: Draw a picture



step 2: Find the limits of integration



step 3: set up and evaluate the integral.

$V = 2 \int_0^{2\pi} \int_0^a \sqrt{a^2 - r^2} r dr d\theta$

recall $dA = r dr d\theta$

no θ in the integrand

$= 2 \cdot 2\pi \int_0^a \sqrt{a^2 - r^2} r dr$

let $u = a^2 - r^2$ $u(0) = a^2$
 $du = -2r dr$ $u(a) = 0$

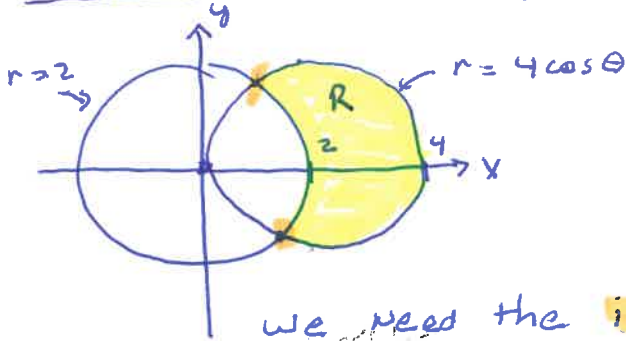
$= 4\pi \int_{a^2}^0 -\frac{1}{2} \sqrt{u} du$

$= -2\pi \left[\frac{2}{3} u^{3/2} \right]_{a^2}^0$

And we see $V = \frac{4}{3} \pi a^3$

setup an integral to represent
 Ex3: $I = \iint_R g(r, \theta) dA$ where R is outside the circle $r = 2$ and inside $r = 4 \cos \theta$

Step 1: sketch a picture



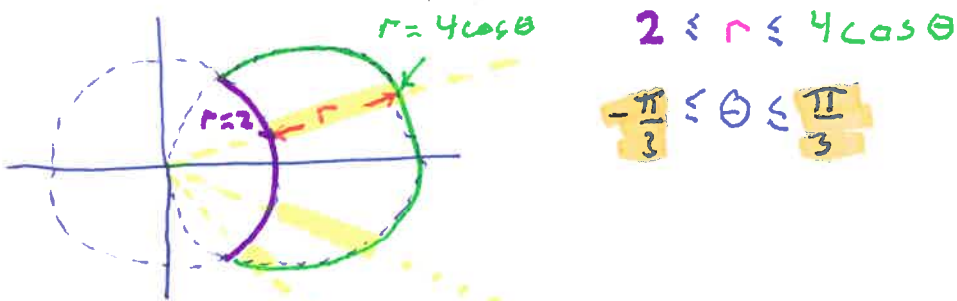
we need the intersection points.

$$\text{solve } 2 = 4 \cos \theta$$

$$\Rightarrow \frac{1}{2} = \cos \theta$$

$$\Rightarrow \theta = \frac{\pi}{3} \text{ or } \theta = -\frac{\pi}{3}$$

Step 2: Find the limits of integration.
 $\theta = \frac{\pi}{3}$ or $\theta = -\frac{\pi}{3}$



Step 3: setup an integral.

$$\begin{aligned}
 I &= \iint_R g(r, \theta) dA \\
 &= \int_{-\pi/3}^{\pi/3} \int_2^{4 \cos \theta} g(r, \theta) r dr d\theta
 \end{aligned}$$



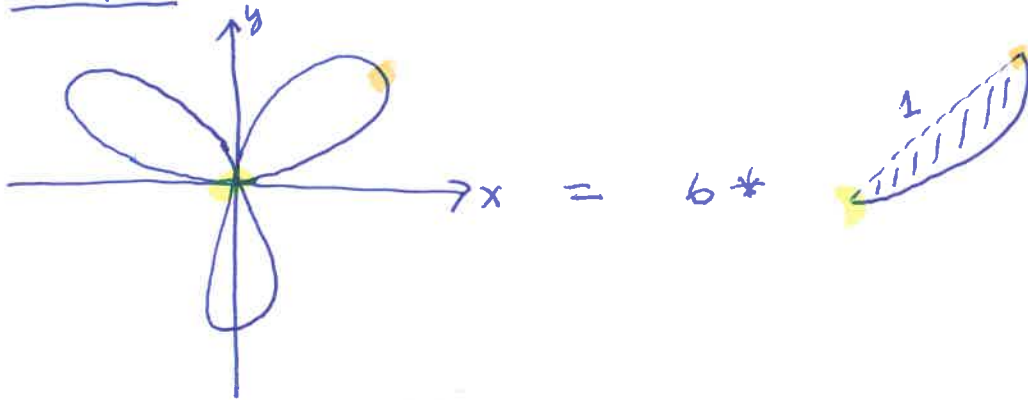
check out
manipulate
16.35

As we saw in calc III, double integrals can be used to find area of a region:

$$\text{area of } D = \iint_D 1 dA$$

Ex4: Use a double polar integral to find the area enclosed by the three-petaled rose $r = \sin(3\theta)$

step 1: sketch the graph,



step 2: Find the limits of integration.

solve: $\sin 3\theta = 0$

$$\Rightarrow \theta = 0, \pm \frac{\pi}{3}, \pm \frac{2\pi}{3}, \pm \frac{3\pi}{3}, \dots$$

solve: $\sin 3\theta = 1$

$$\Rightarrow 3\theta = \frac{\pi}{2} + 2k\pi$$

$$\Rightarrow \theta = \frac{\pi}{6} + \frac{2}{3}k\pi$$

CAUTION! THERE IS A SUBTLE PITFALL HERE.

step 3: setup and evaluate an integral to represent the area.

$$\begin{aligned} A &= 6 \int_0^{\pi/6} \int_0^{\sin 3\theta} 1 r dr d\theta \\ &= 6 \int_0^{\pi/6} \left[\frac{r^2}{2} \right]_0^{\sin 3\theta} d\theta \\ &= 3 \int_0^{\pi/6} \sin^2(3\theta) d\theta \end{aligned} \quad \rightarrow \quad \begin{aligned} A &= 3 \int_0^{\pi/6} \frac{1 - \cos(6\theta)}{2} d\theta \\ &= \frac{3}{2} \left[\theta - \frac{1}{6} \sin(6\theta) \right]_0^{\pi/6} \\ &= \frac{3}{2} \cdot \frac{\pi}{6} \\ &= \frac{\pi}{4} \end{aligned}$$

This is the area of the polar rose.

Historical anecdote: There is a famous story about the nineteenth-century Scottish physicist Lord Kelvin. "Do you know what a mathematician is?" Kelvin once asked a class. He stepped to the blackboard and wrote:

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$

"A mathematician," he continued, "is one to whom that is as obvious as $2 \cdot 2 = 4$ is to you." Let's explore just how obvious this really is ...

Ex5: evaluate $I = \int_{-\infty}^{\infty} e^{-x^2} dx$

$$\begin{aligned}
 I &= \int_{-\infty}^{\infty} e^{-x^2} dx \\
 \Rightarrow I &= 2 \int_0^{\infty} e^{-x^2} dx \\
 \Rightarrow I^2 &= 4 \left(\int_0^{\infty} e^{-x^2} dx \right)^2 \\
 &= 4 \int_0^{\infty} e^{-x^2} dx \int_0^{\infty} e^{-y^2} dy \\
 &= 4 \int_0^{\infty} \int_0^{\infty} e^{-(x^2+y^2)} dx dy \\
 &= 4 \int_0^{\frac{\pi}{2}} \int_0^{\infty} e^{-r^2} r dr d\theta \\
 &= \frac{4}{2} \cdot \frac{\pi}{2} \left[e^{-r^2} \right]_0^{\infty} \\
 &\rightarrow = -\pi (0 - 1) \\
 &= \pi \\
 \text{Thus } I^2 &= \pi \\
 \text{and } I &= \int_{-\infty}^{\infty} e^{-x^2} = \sqrt{\pi} \\
 &\text{as Kelvin claimed.}
 \end{aligned}$$

Reflection on the anecdote: After working through this example it is clear that this formula is *not* obvious to your teacher or anyone who he knows. The conclusion seems to be that Kelvin was both showing off and trying to put down his class in a rather mean-spirited way.

Application: This integrand is called the normal or Gaussian distribution and is important in probability

theory. The general form is $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$ where μ represents the mean and σ the

standard deviation of a distribution. Some examples of (roughly) normally distributed variables include height, rolling a dice, tossing a coin, IQ, technical stock market, income distribution, shoe size, birth weight, and student grades.