

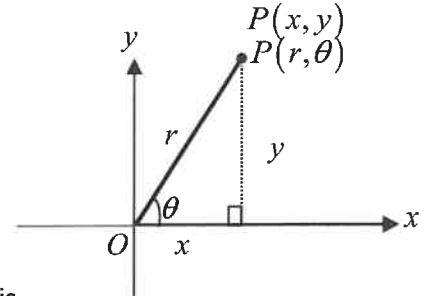
HW: 30 min + 5 Q's  
54 min + 5 Q's

**Polar Coordinates**

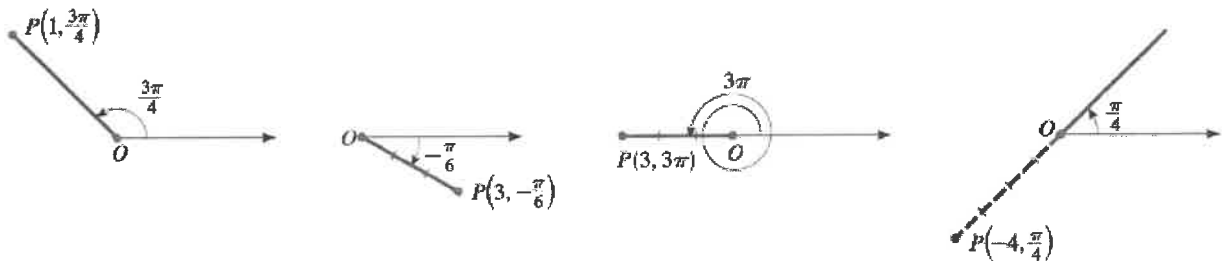
We've been using the Cartesian (rectangular) coordinate system to graph. Another useful coordinate system is the **polar coordinate system** which uses distance and direction to specify the location of a point in the plane. The system is based on a point, called the **pole** (or origin) and a ray drawn in the direction of the x-axis, called the **polar axis**.

$r$  is the distance from  $O$  to  $P$ .

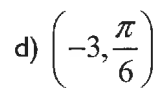
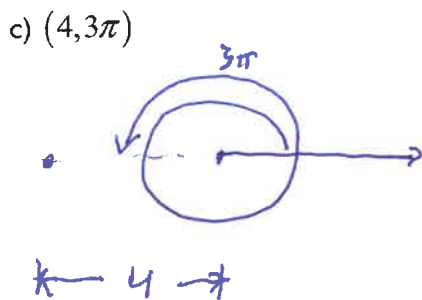
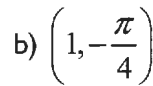
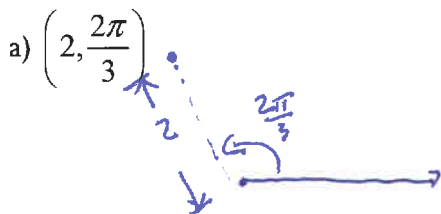
$\theta$  is the angle between the polar axis and the segment  $\overline{OP}$ .



$\theta$  is positive if measured in a counterclockwise direction from the polar axis or negative if measured in a clockwise direction. If  $r$  is negative, then  $P$  is the point that is  $|r|$  units from the pole in the direction opposite to that given by  $\theta$ .

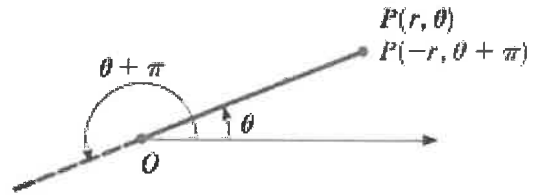


**Ex1:** Plot the given points.



Note: The coordinates  $(r, \theta)$  and  $(-r, \theta + \pi)$  represent the same point. Also keep in mind that because of the coterminal angles, each point has infinitely many representations.  $P(r, \theta)$  is the same as:

$$P(r, \theta + 2n\pi) \quad \text{and} \quad P(-r, \theta + (2n+1)\pi)$$



Ex2: Graph  $(1, \frac{5\pi}{4})$  Find two other polar coordinate representations with  $r > 0$ , and two with  $r < 0$ .

$$\begin{aligned} & (1, \frac{5\pi}{4}) \\ & (1, \frac{5\pi}{4} + 2\pi) & (-1, \frac{5\pi}{4} - \pi) \\ & (1, \frac{5\pi}{4} + 4\pi) & (-1, \frac{5\pi}{4} + \pi) \end{aligned}$$



check out  
manipulate  
12, 20

### RELATIONSHIP BETWEEN POLAR AND RECTANGULAR COORDINATES

1. To change from polar to rectangular coordinates, use the formulas

$$x = r \cos \theta \quad \text{and} \quad y = r \sin \theta$$

2. To change from rectangular to polar coordinates, use the formulas

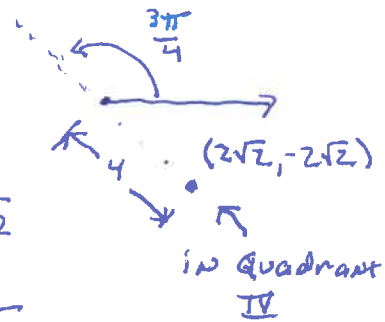
$$r^2 = x^2 + y^2 \quad \text{and} \quad \tan \theta = \frac{y}{x} \quad (x \neq 0)$$

Note: These equations do not uniquely determine  $r$  or  $\theta$ . Check to make sure you are in the correct quadrant.

Ex3: Find the rectangular coordinates of the point  $(-4, \frac{3\pi}{4})$ .

$$x = -4 \cos \frac{3\pi}{4} = -4 \left( -\frac{\sqrt{2}}{2} \right) = 2\sqrt{2}$$

$$y = -4 \sin \frac{3\pi}{4} = -4 \left( \frac{\sqrt{2}}{2} \right) = -2\sqrt{2}$$



The quadrant is right correct from polar to rectangular.

**Ex4:** Find the polar coordinates of the point  $(-1, -\sqrt{3}) \leftarrow Q3$

$$r = \sqrt{(-1)^2 + (\sqrt{3})^2} = 2$$

$$\theta = \tan^{-1}\left(\frac{-\sqrt{3}}{-1}\right) = \frac{\pi}{3}$$

so the point is  $(2, \frac{4\pi}{3})$

so at first glance the coordinates are  $(2, \frac{\pi}{3})$  which is in **quadrant 1**. But we see that tangent  $> 0$  in **Q1 & Q3**

double check the quadrant when going from rect. to polar

$$\frac{\pi}{3} + \pi$$

**Ex5:** Express the following equations in polar coordinates.

a)  $x = 2$

$$2 = r \cos \theta$$

$$\Rightarrow r = 2 \sec \theta$$

b)  $x^2 + y^2 = 9$

$$r^2 = 9$$

$$\Rightarrow r = 3$$

**Ex6:** Express the following equations in rectangular coordinates.

a)  $r = \frac{4}{1 + \sin \theta}$

$$r = \frac{4}{1 + \sin \theta}$$

$$= \frac{4}{1 + \frac{y}{r}}$$

$$= \frac{4r}{r + y}$$

$$\Rightarrow r(r + y) = 4r$$

$$\Rightarrow r + y = 4$$

$$\Rightarrow (y - 4)^2 = (-r)^2$$

$$\Rightarrow y^2 - 8y + 16 = r^2$$

$$= x^2 + y^2$$

$$\Rightarrow -8y + 16 = x^2$$

$$\Rightarrow -8y = x^2 - 16$$

$$\Rightarrow y = -\frac{1}{8}x^2 + 2$$

b)  $\tan \theta = 1$

$\tan \theta = 1$

$\Rightarrow \frac{y}{x} = 1$

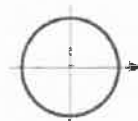
$\Rightarrow y = x$

❖ **Polar Curves**

The **graph of a polar equation**  $r = f(\theta)$ , or more generally  $F(r, \theta) = 0$ , consists of all points  $P$  that have at least one polar representation  $(r, \theta)$  whose coordinates satisfy the equation.

**SOME COMMON POLAR CURVES**

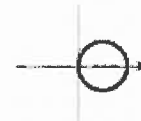
**Circles and Spiral**



$r = a$   
circle



$r = a \sin \theta$   
circle



$r = a \cos \theta$   
circle



$r = a\theta$   
spiral

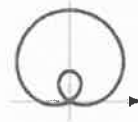
**Limaçons**

$r = a \pm b \sin \theta$

$r = a \pm b \cos \theta$

$(a > 0, b > 0)$

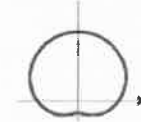
Orientation depends on the trigonometric function (sine or cosine) and the sign of  $b$ .



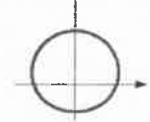
$a < b$   
limaçon with inner loop



$a = b$   
cardioid



$a > b$   
dimpled limaçon



$a \geq 2b$   
convex limaçon

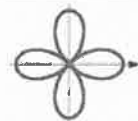
**Roses**

$r = a \sin n\theta$

$r = a \cos n\theta$

$n$ -leaved if  $n$  is odd

$2n$ -leaved if  $n$  is even



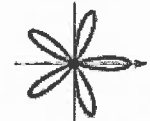
$r = a \cos 2\theta$   
4-leaved rose



$r = a \cos 3\theta$   
3-leaved rose



$r = a \cos 4\theta$   
8-leaved rose



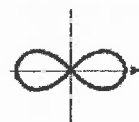
$r = a \cos 5\theta$   
5-leaved rose

**Lemniscates**

Figure-eight-shaped curves



$r^2 = a^2 \sin 2\theta$   
lemniscate



$r^2 = a^2 \cos 2\theta$   
lemniscate

❖ **Tangents to Polar Curves**

To find a tangent line to a polar curve  $r = f(\theta)$ , we regard  $\theta$  as a parameter and write its parametric equations as

$$x = r \cos \theta = f(\theta) \cos \theta \quad y = r \sin \theta = f(\theta) \sin \theta$$

If given  $(x(t), y(t))$   
then  $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$

Then, using the method for finding slopes of parametric curves (from calc III) and the **Product Rule**, we have

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{d}{d\theta}(r \sin \theta)}{\frac{d}{d\theta}(r \cos \theta)} = \frac{\frac{dr}{d\theta} \sin \theta + r \cos \theta}{\frac{dr}{d\theta} \cos \theta - r \sin \theta}$$

Note:  $\frac{dy}{dx} \neq \frac{df}{d\theta}$

Notice that if we are looking for tangent lines at the pole, then  $r = 0$  and this formula simplifies to:

$$\frac{dy}{dx} = \tan \theta \quad \text{if } \frac{dr}{d\theta} \neq 0$$

**Ex9:** Find the slope of the tangent line to  $r = 1 - \cos \theta$  at  $\theta$ .

$$r = 1 - \cos \theta$$

$$\frac{dr}{d\theta} = \sin \theta$$

so using the formula  $\frac{dy}{dx} = \frac{r' \sin \theta + r \cos \theta}{r' \cos \theta - r \sin \theta}$

$$\text{we have } \frac{dy}{dx} = \frac{\sin \theta \cdot \sin \theta + (1 - \cos \theta) \cos \theta}{\sin \theta \cos \theta - (1 - \cos \theta) \sin \theta}$$

$$= \frac{\sin^2 \theta + \cos \theta - \cos^2 \theta}{\sin \theta \cos \theta - \sin \theta + \sin \theta \cos \theta}$$

$$= \frac{\cos \theta - \cos^2 \theta}{\sin 2\theta - \sin \theta}$$

$$= \frac{\cos \theta - \cos^2 \theta}{\sin 2\theta - \sin \theta}$$



check out  
manipulate  
12.35

check out  
manipulate  
12,38



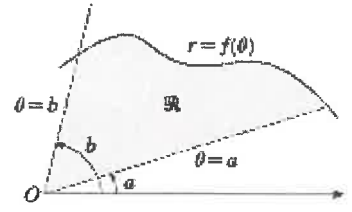
when  $\theta = 2\pi$  this becomes  $A = \frac{1}{2} r^2 \cdot 2\pi$  or  $A = \pi r^2$  which is the area of a circle.



❖ Area in Polar Coordinates

Recall <sup>the</sup> area of a sector:  $A = \frac{1}{2} r^2 \theta$

Now suppose that we want to find the area of the shaded region on the given graph where  $f$  is positive and continuous and  $0 < b - a \leq 2\pi$ .

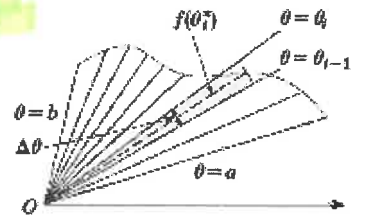


Divide  $[a, b]$  into subintervals of equal width  $\Delta\theta$ . Then:

$\Delta A_i \approx \frac{1}{2} [f(\theta_i^*)]^2 \Delta\theta$  ←  $\Delta A = \frac{1}{2} r^2 \Delta\theta$  with  $r$  replaced by  $f(\theta_i^*)$

$A \approx \sum_{i=1}^n \frac{1}{2} [f(\theta_i^*)]^2 \Delta\theta$

$A = \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{1}{2} [f(\theta_i^*)]^2 \Delta\theta = \int_a^b \frac{1}{2} [f(\theta)]^2 d\theta$

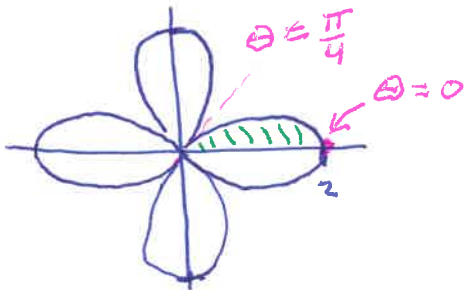


This is often written as:

$$A = \int_a^b \frac{1}{2} r^2 d\theta$$

**Ex 10:** Find the area of the region in the plane enclosed by the polar rose  $r = 2 \cos 2\theta$

step 1: sketch the region. step 3: set-up the integral and integrate.



step 2: Find the limits of integration.

solve  $0 = 2 \cos 2\theta$   
 $\Rightarrow 0 = \cos 2\theta$   
 $\Rightarrow \theta = \frac{\pi}{4} + k \frac{\pi}{2}$   
 solve  $2 = 2 \cos 2\theta$   
 $\Rightarrow \theta = 0 + 2k\pi$

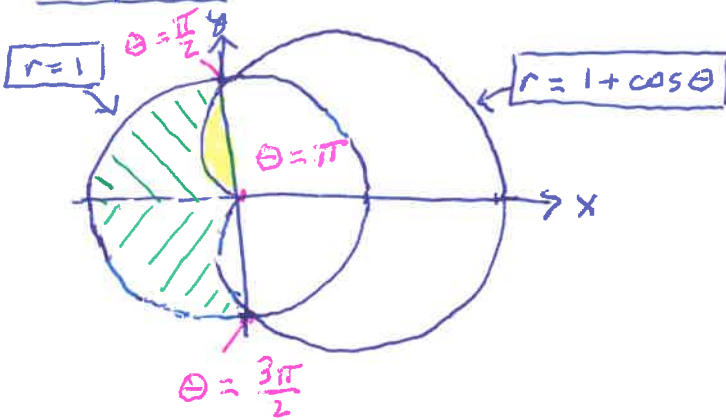
$$\begin{aligned} A &= 8 \cdot \frac{1}{2} \int_0^{\pi/4} (2 \cos 2\theta)^2 d\theta \\ &= 16 \int_0^{\pi/4} \cos^2 2\theta d\theta \\ &= 16 \int_0^{\pi/4} \frac{1 + \cos 4\theta}{2} d\theta \\ &= 8 \left[ \theta + \frac{1}{4} \sin 4\theta \right]_0^{\pi/4} \\ &= 2\pi \end{aligned}$$



check out  
manipulate  
12,38

**Ex 11:** Find the area of the region that lies inside the circle  $r=1$  and outside the cardioid  $r=1+\cos\theta$ .

Step 1: sketch the region



Step 2: find the limits of integration

solve  $1 = 1 + \cos\theta$

$\Rightarrow 0 = \cos\theta$

$\Rightarrow \theta = \frac{\pi}{2} + k\pi$

solve  $0 = 1 + \cos\theta$

$\Rightarrow -1 = \cos\theta$

$\Rightarrow \theta = \pi + 2k\pi$

Step 3: set-up the integral and integrate.

$$\begin{aligned} \frac{1}{2} A &= \left[ \frac{\pi}{4} \right] - \left[ \text{yellow region} \right] \\ &= \frac{\pi}{4} - \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{1}{2} (1 + \cos\theta)^2 d\theta \\ &= \frac{\pi}{4} - \frac{1}{2} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} 1 + 2\cos\theta + \cos^2\theta d\theta \\ &= \frac{\pi}{4} - \frac{1}{2} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} 1 + 2\cos\theta + \frac{1 + \cos 2\theta}{2} d\theta \\ &= \frac{\pi}{4} - \frac{1}{2} \left[ \frac{3}{2}\theta + 2\sin\theta + \frac{\sin 2\theta}{4} \right]_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \\ &= \frac{\pi}{4} - \left( \frac{3}{8}\pi - 1 \right) \end{aligned}$$

$\Rightarrow \frac{1}{2} A = 1 - \frac{\pi}{8}$

AND  $A = 2 - \frac{\pi}{4}$



check out  
manipulate  
12.40

❖ **Length of the Polar Curves**

To find the length of a polar curve  $r = f(\theta)$ ,  $a \leq \theta \leq b$ , we regard  $\theta$  as a parameter and write the parametric equations of the curve as

$x = r \cos\theta = f(\theta) \cos\theta$        $y = r \sin\theta = f(\theta) \sin\theta$

Using the Product Rule and differentiating with respect to  $\theta$ , we obtain

$\frac{dx}{d\theta} = \frac{dr}{d\theta} \cos\theta - r \sin\theta$        $\frac{dy}{d\theta} = \frac{dr}{d\theta} \sin\theta + r \cos\theta$

If given  $(x(t), y(t))$   
then the arclength  
is found with  
 $L = \int_{t=a}^{t=b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$   
(from calc III)

$$\begin{aligned} \left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 &= \left(\frac{dr}{d\theta}\right)^2 \cos^2\theta - 2r \frac{dr}{d\theta} \cos\theta \sin\theta + r^2 \sin^2\theta \\ &\quad + \left(\frac{dr}{d\theta}\right)^2 \sin^2\theta + 2r \frac{dr}{d\theta} \sin\theta \cos\theta + r^2 \cos^2\theta \\ &= \left(\frac{dr}{d\theta}\right)^2 + r^2 \end{aligned}$$

Assuming that  $f'$  is continuous, we can use our theorem from calc III to write:

$$L = \int_a^b \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} d\theta$$

Therefore the length of a curve with polar equation  $r = f(\theta)$ ,  $a \leq \theta \leq b$ , is

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$$

**Ex12:** Find the length of the cardioid  $r = 1 + \cos\theta$ .

$$r = 1 + \cos\theta \quad \text{and} \quad \frac{dr}{d\theta} = -\sin\theta$$

$$\Rightarrow L = 2 \int_0^\pi \sqrt{(1 + \cos\theta)^2 + (-\sin\theta)^2} d\theta$$

symmetry

$$= 2 \int_0^\pi \sqrt{1 + 2\cos\theta + \underbrace{\cos^2\theta + \sin^2\theta}_1} d\theta$$

$$= 2 \int_0^\pi \sqrt{\underbrace{2 + 2\cos\theta}_{4\left(\frac{1+\cos\theta}{2}\right)}} d\theta \rightarrow L = 8 \left[ \sin\frac{\theta}{2} \right]_0^\pi = 8 \text{ (arclength)}$$

$$= 2 \cdot 2 \int_0^\pi \sqrt{\cos^2\left(\frac{\theta}{2}\right)} d\theta$$

$$= 4 \int_0^\pi \cos\left(\frac{\theta}{2}\right) d\theta$$