Strategy for Testing Series

Let's do a brief review of THE LIST all the tests we have introduced for determining convergence and how to decide which test to use.

To do this, remember the 4 known series:

- Geometric series: $\sum_{n=0}^{\infty} ar^n \begin{cases} converges & if \quad |r|<1 \\ diverges & if \quad |r|\geq 1 \end{cases}$
- P-series: $\sum_{n=1}^{\infty} \frac{1}{n^p} \begin{cases} converges & if \quad p>1 \\ diverges & if \quad p\leq 1 \end{cases}$

The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges

The alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges (conditionally)

Let $\sum_{n=1}^{\infty} a_n$ be given.

For Positive Series:

(easy weak) Test for Divergence

- **The Divergent Test (nth-Term Test):** Always check this test first.

$$\lim_{n\to\infty} a_n \begin{cases} =0 & then & inconclusive! \\ \neq 0 & then & \sum_{n=1}^{\infty} a_n \quad diverges \end{cases}$$

(hard powerful :)

- **The Direct Comparison Test:** Consider whether dropping terms in the numerator or denominator gives a series that we know converges or diverges, $\sum_{n=1}^{\infty} b_n$.

$\sum b_n$ known

Pro-tip: If using a comparison test, remember that you need to specifically state whether the series you are comparing to converges or diverges.

$$If \quad a_n \leq b_n \quad and \quad \sum_{n=1}^{\infty} b_n \quad converges \quad then \quad \sum_{n=1}^{\infty} a_n \quad converges$$

$$If \quad a_n \geq b_n \quad and \quad \sum_{n=1}^{\infty} b_n \quad diverges \quad then \quad \sum_{n=1}^{\infty} a_n \quad diverges$$

	Bigger	Smaller
converge	$\sum a_n$ converges	inconclusive
diverge	inconclusive	$\sum a_n$ diverges

(strong easy)

- **The Limit Comparison Test:** Consider the dominant term in the numerator and denominator, and come up with a series that we know converges or diverges, $\sum_{n=1}^{\infty} b_n$.

  - If $\lim_{n\to\infty} \frac{a_n}{b_n} \begin{cases} >0 \\ finite \end{cases}$, then either $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge or both diverge.
  - If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.
  - If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

$\sum b_n$	known
converges	$\sum a_n$ converges
diverges	$\sum a_n$ diverges

Don't forget the form that the conclusion should take: Therefore the series ___ by the ___ test.

(easy - strong)

- **The Ratio Test.** Best to use when there is a factorial or a constant to the power of n. Don't use w/ p-series.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|\begin{cases}<1 & then \quad absolutely-convergent \Rightarrow convergent\\ >1 & then \quad divergent\\ =1 & then \quad inconclusive!\end{cases}$$

(easy - weak)

- **The Root Test.** Best to use when the terms have power n. Don't use w/ p-series.

$$\lim_{n\to\infty}\sqrt[n]{|a_n|}\begin{cases}<1 & then \quad absolutely-convergent \Rightarrow convergent\\ >1 & then \quad divergent\\ =\infty & then \quad divergent\\ =1 & then \quad inconclusive!\end{cases}$$

(hard - weak)

- **The Integral Test.** Best to use when the other tests fail. If $a_n = f(n)$ is decreasing, positive and continuous,

$$\int_1^{\infty} f(x)\,dx \begin{cases} converges & then \quad \sum_{n=1}^{\infty} a_n \quad converges\\ diverges & then \quad \sum_{n=1}^{\infty} a_n \quad diverges\end{cases}$$

(hard - weak).

- Telescoping series. Use this when the terms can be written as a difference where terms cancel. The examples often require basic partial fractions, log rules, or trig identities.

For Series That Are Not Positive Series:

- **Alternating Series Test.** For series of the form $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}(-1)^{n-1} b_n$

$$If \begin{cases} b_{n+1} \le b_n \\ \lim_{n\to\infty} b_n = 0\end{cases} \quad then \quad \sum_{n=1}^{\infty} a_n$$ converges (either absolutely or conditionally).

Reminder: **Absolute Convergence.** If $\sum_{n=1}^{\infty} a_n$ has some negative terms (maybe alternating or not), then consider $\sum_{n=1}^{\infty}|a_n|$. If $\sum_{n=1}^{\infty}|a_n|$ converges, $\sum_{n=1}^{\infty} a_n$ is converging absolutely.

In short:

1. **The nth-Term Test**: Unless $a_n \to 0$, the series diverges.
2. **Geometric series**: $\sum ar^n$ converges if $|r|<1$; otherwise it diverges.
3. **p-series**: $\sum \frac{1}{n^p}$ converges if $p>1$; otherwise it diverges.
4. **Series with nonnegative terms**: Try the Integral Test, Ratio Test or Root Test. Try comparing to a known series with the Comparison Test or the Limit Comparison test.
5. **Series with some negative terms**: Does $\sum |a_n|$ converges? If yes, so does $\sum a_n$ since absolute convergence implies convergence.
6. **Alternating series**: $\sum a_n$ converges if $\sum |a_n|$ is decreasing and $|a_n| \to 0$.

	EASY	HARD
WEAK	Test for Divergence Alternating Series Test Harmonic series Alt. Harmonic Series Geometric series p- series	Telescoping series
POWERFUL	Limit comparison test Ratio Test Root Test	Comparison. Test Integral Test