

The Comparison Tests

The Comparison Test Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

- (i) If $\sum b_n$ is convergent and $a_n \leq b_n$ for all n , then $\sum a_n$ is also convergent.
 (ii) If $\sum b_n$ is divergent and $a_n \geq b_n$ for all n , then $\sum a_n$ is also divergent.

In using the Comparison Test we must, of course, have some known series $\sum b_n$ for the purpose of comparison. Most of the time we use one of these series:

- A p -series [$\sum 1/n^p$ converges if $p > 1$ and diverges if $p \leq 1$];
- A geometric series [$\sum ar^{n-1}$ converges if $|r| < 1$ and diverges if $|r| \geq 1$];

Ex 1: Determine the convergence of the following.

a) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}3^n} \leq \sum_{n=1}^{\infty} \frac{1}{3^n}$ which is a convergent geometric series.

$\therefore \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}3^n}$ converges by the comparison test.

Form of conclusion: Therefore the series ____ by the ____ test.

b) $\sum_{n=1}^{\infty} \frac{1}{(n^2+3)^{1/3}} \leq \sum_{n=1}^{\infty} \frac{1}{n^{2/3}}$ Divergent p-series.

The test is inconclusive.

visualize this with:

known series

	BIGGER	SMALLER
CONVERGES	UNKNOWN CONVERGES	?
DIVERGES	?	UNKNOWN DIVERGES

THE LIST: (7) The comparison test (weak)

NOTE 1 Although the condition $a_n \leq b_n$ or $a_n \geq b_n$ in the Comparison Test is given for all n , we need verify only that it holds for $n \geq N$, where N is some fixed integer, because the convergence of a series is not affected by a finite number of terms.

Ex2: Determine the convergence of the following.

$$\sum_{n=1}^{\infty} \frac{n^2}{n^4 + n - 10} = \sum_{n=1}^{10} \frac{n^2}{n^4 + n - 10} + \sum_{n=11}^{\infty} \frac{n^2}{n^4 + n - 10}$$

Notice the explicit mention of the convergence of the p-series.

$$< \quad \quad \quad + \sum_{n=11}^{\infty} \frac{1}{n^2} \quad \text{a convergent p-series.}$$

∴ The series converges by the comparison test.

Form of conclusion: Therefore the series ____ by the ____ test.

NOTE 2 The terms of the series being tested must be smaller than those of a convergent series or larger than those of a divergent series. If the terms are larger than the terms of a convergent series or smaller than those of a divergent series, then the Comparison Test doesn't apply.

To be able to answer the last problem we use the following variation of the Direct Comparison test.

The Limit Comparison Test Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms. If

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = c$$

where c is a finite number and $c > 0$, then either both series converge or both diverge.

A couple special cases:

- If $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 0$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.
- If $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \infty$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

Back to Ex2: $\sum_{n=1}^{\infty} \frac{n^2}{n^4 + n - 10}$

consider $\lim_{n \rightarrow \infty} \frac{\frac{1}{n^2}}{\frac{n^4 + n - 10}{n^4}} = \lim_{n \rightarrow \infty} \frac{n^4 - n - 1}{n^4} = 1$

Notice the explicit mention that the series we are comparing with converges.

Since the limit is a number the two series share the same fate.

\therefore since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (a p-series)

we know $\sum_{n=1}^{\infty} \frac{n^2}{n^4 + n - 10}$ converges

by the limit comparison test,

Back to Ex1b: $\sum_{n=1}^{\infty} \frac{1}{(n^2 + 3)^{1/3}}$

consider $\lim_{n \rightarrow \infty} \frac{1}{(n^2 + 3)^{1/3}} \cdot \frac{n^{2/3}}{n^{2/3}} = \lim_{n \rightarrow \infty} \frac{n^{2/3}}{(n^2 + 3)^{1/3}} = 1$

\therefore since $\sum_{n=1}^{\infty} \frac{1}{n^{2/3}}$ diverges (a p-series) we know

$\sum_{n=1}^{\infty} \frac{1}{(n^2 + 3)^{1/3}}$ diverges by the limit comparison test.