

## Change of Variables in Multiple Integrals

Objective:

1. Change of variable in a single integral -  $u$  substitution
2. Where might this have previously helped and examples
3. The Theory
4. Examples continued

In this section we will discuss a general method of evaluating double and triple integrals by substitution. Most of the results in this section are very difficult to prove, so our approach will be informal and motivational. Our goal is to provide a geometric understanding of the basic principles and an exposure to computational techniques.

### *I. Change of Variable in a Single Integral*

Ex0:  $I = \int_a^b \sin(2x) dx$

The main point: Substitution requires changing the integrand, limits of integration, and the introduction of a scaling factor.

**2. Where might this have previously helped and examples**

One of the homework/assessment questions is an excellent motivation for a change of variables (substitution).

Ex1: Use a triple integral to compute the volume of the wedge of the square column  $|x| + |y| = 10$  created by the planes  $z = 0$  and  $x + y + z = 10$ .

The following examples allow you to focus on the impact of a multivariate substitution on the integrand and limits of integration. At this point, we will not seek out the scaling factor.

Ex2: Set up an iterated integral to represent  $I = \iint_D \sin(9x^2 + 4y^2) dA$  where  $D$  is the region in the first quadrant bounded by  $9x^2 + 4y^2 = 1$

Ex3: Set up an iterated integral to represent  $I = \iint_D \frac{x-2y}{3x-y} dA$  where  $D$  is the parallelogram enclosed by  $x-2y=0$ ;  $x-2y=4$ ;  $3x-y=1$ ; and  $3x-y=8$ .

In order for a multivariate substitution to be helpful, the substitution must “help” both integrand AND limits of integration.

Ex4: Set up an iterated integral to represent  $I = \iint_D (x+y)e^{x^2-y^2} dA$  where  $D$  is the region enclosed by  $x+y=0$ ;  $x+y=3=4$ ;  $x-y=0$ ; and  $x-y=2$ .

The scaling factor = | “The Jacobian” |

Ex1 revisited: Use a triple integral to compute the volume of the wedge of the square column  $|x| + |y| = 10$  created by the planes  $z = 0$  and  $x + y + z = 10$ .

### 3. The Theory

To derive the change-of-variable formula for double integrals, we need to understand the relationship between the area of a small rectangular region in the  $uv$ -plane and the area of its image in the  $xy$ -plane under a transformation  $T$  given by equations  $x = x(u, v)$  and  $y = y(u, v)$ .

For this purpose, suppose that  $\Delta u$  and  $\Delta v$  are positive and small, and consider a rectangular region  $S$  in the  $uv$ -plane enclosed by the lines

$$u = u_0, \quad u = u_0 + \Delta u, \quad v = v_0, \quad v = v_0 + \Delta v$$

Then  $R$  in  $xy$ -plane is its image. Let

$$\vec{r}(u, v) = \langle x(u, v), y(u, v) \rangle$$

be the position vector to the point in  $xy$ -plane

that corresponds to the point  $(u, v)$  in the  $uv$ -plane, then

$$\vec{r}(u, v_0) = \langle x(u, v_0), y(u, v_0) \rangle \text{ corresponds to } v = v_0$$

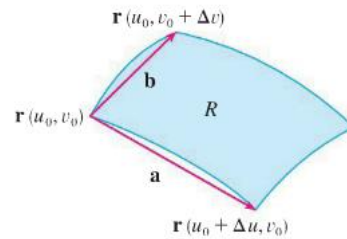
and

$$\vec{r}(u_0, v) = \langle x(u_0, v), y(u_0, v) \rangle \text{ corresponds to } u = u_0.$$

Since we assumed that  $\Delta u$  and  $\Delta v$  are small, the region  $R$  can be approximated by a parallelogram determined by the secant vectors:

$$\vec{a} = \vec{r}(u_0 + \Delta u, v_0) - \vec{r}(u_0, v_0)$$

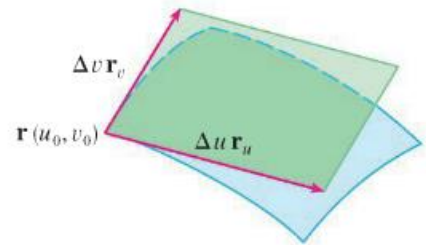
$$\vec{b} = \vec{r}(u_0, v_0 + \Delta v) - \vec{r}(u_0, v_0)$$



A more useful approximation of  $R$  can be obtained by using tangent vectors as follows:

$$\vec{a} = \frac{\vec{r}(u_0 + \Delta u, v_0) - \vec{r}(u_0, v_0)}{\Delta u} \Delta u \approx \frac{\partial \vec{r}}{\partial u} \Delta u = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u} \right\rangle \Delta u = \vec{r}_u \Delta u$$

$$\vec{b} = \frac{\vec{r}(u_0, v_0 + \Delta v) - \vec{r}(u_0, v_0)}{\Delta v} \Delta v \approx \frac{\partial \vec{r}}{\partial v} \Delta v = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v} \right\rangle \Delta v = \vec{r}_v \Delta v$$



where the partial derivatives are evaluated at  $(u_0, v_0)$ .

At  $T(u_0, v_0) = (x_0, y_0)$  the vectors  $\vec{r}_u$  and  $\vec{r}_v$  are tangent to  $\vec{r}(u, v_0)$  and  $\vec{r}(u_0, v)$  respectively, and so are  $\vec{r}_u \Delta u$  and  $\vec{r}_v \Delta v$ . Thus the area of  $R$  which we denote by  $\Delta A$  can be approximated by the area of the parallelogram determined by these vectors.

$$\Delta A \approx \left| \vec{r}_u \Delta u \times \vec{r}_v \Delta v \right| = \left| \vec{r}_u \times \vec{r}_v \right| \Delta u \Delta v$$

where the partial derivatives are evaluated at  $(u_0, v_0)$ .

Computing the cross product, we obtain

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & 0 \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & 0 \end{vmatrix} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} \mathbf{k} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \mathbf{k}$$

This determinant is so important that it has its own terminology!

**Definition** The **Jacobian** of the transformation  $T$  given by  $x = g(u, v)$  and  $y = h(u, v)$  is

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u}$$

Therefore: 
$$\vec{r}_u \times \vec{r}_v = \frac{\partial(x, y)}{\partial(u, v)} \vec{k}$$

So we can write the area as:

$$\Delta A \approx \left| \frac{\partial(x, y)}{\partial(u, v)} \vec{k} \right| \Delta u \Delta v$$

Since  $\vec{k}$  is a unit vector, the magnitude of  $\frac{\partial(x, y)}{\partial(u, v)} \vec{k}$  will be the absolute value of  $\frac{\partial(x, y)}{\partial(u, v)}$ :

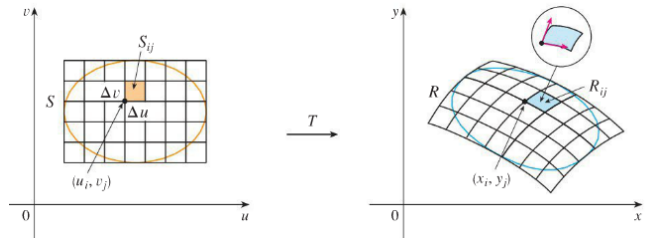
$$\left| \frac{\partial(x, y)}{\partial(u, v)} \vec{k} \right| = \left| \frac{\partial(x, y)}{\partial(u, v)} \right|$$

Hence: 
$$\Delta A \approx \left| \frac{\partial(x, y)}{\partial(u, v)} \right| \Delta u \Delta v$$

At the point  $(u_0, v_0)$  this important formula relates the area of the region  $R$  and  $S$ . It tells us that for small values of  $\Delta u$  and  $\Delta v$ , the area of  $R$  is approximately the absolute value of the Jacobian times the area of  $S$ . In advanced calculus it's proven that this error approaches zero as  $\Delta u$  and  $\Delta v$  approach zero.

Now let's look at double integrals by dividing the regions into small rectangles and apply the Riemann sum:

$$\begin{aligned} \iint_R f(x, y) dA &\approx \sum_{i=1}^m \sum_{j=1}^n f(x_i, y_j) \Delta A \\ &\approx \sum_{i=1}^m \sum_{j=1}^n f(x_i, y_j) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| \Delta u \Delta v \end{aligned}$$



where  $x_i, y_j$  and the Jacobian are evaluated at  $(u_i, v_j)$ . This is the approximation for

$\iint_S f(u, v) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| \Delta u \Delta v$  so the following theorem can be used:

**Change of Variables in a Double Integral** Suppose that  $T$  is a  $C^1$  transformation whose Jacobian is nonzero and that  $T$  maps a region  $S$  in the  $uv$ -plane onto a region  $R$  in the  $xy$ -plane. Suppose that  $f$  is continuous on  $R$  and that  $R$  and  $S$  are type I or type II plane regions. Suppose also that  $T$  is one-to-one, except perhaps on the boundary of  $S$ . Then

$$\iint_R f(x, y) dA = \iint_S f(x(u, v), y(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du dv$$

This theorem says that we change from an integral in of  $x$  and  $y$  to an integral in of  $u$  and  $v$  by

expressing  $x$  and  $y$  in terms of  $u$  and  $v$  and writing  $dA = \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du dv$ .

**4. Examples continued**

Ex2 revisited: Evaluate  $I = \iint_D \sin(9x^2 + 4y^2) dA$  where  $D$  is the region in the first quadrant bounded by

$$9x^2 + 4y^2 = 1$$



Ex3 revisited: Set up an iterated integral to represent  $I = \iint_D \frac{x-2y}{3x-y} dA$  where  $D$  is the parallelogram enclosed by  $x-2y=0$ ;  $x-2y=4$ ;  $3x-y=1$ ; and  $3x-y=8$ .

**Ex4 revisited:** Evaluate  $I = \iint_D (x+y)e^{x^2-y^2} dA$  where  $D$  is the region enclosed by  $x+y=0$ ;

$x+y=3$ ;  $x-y=0$ ; and  $x-y=2$ .

Connection: What is the scaling factor when changing from rectangular  $(x, y)$  coordinates to polar  $(r, \theta)$  coordinates?

Connection: What is the scaling factor when changing from rectangular  $(x, y, z)$  coordinates to spherical  $(\rho, \phi, \theta)$  coordinates?