Series

An **infinite series** is the sum of an infinite sequence of numbers

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \ldots + a_n + \ldots$$

The goal of this section is to understand the meaning of such an infinite sum and to develop methods of calculation. You might assume adding infinitely many numbers always give you infinity but that is not necessarily true! For example:

What do we mean when we express a number as an infinite decimal? For instance, what does it mean to write

$$\pi = 3.14159\ 26535\ 89793\ 23846\ 26433\ 83279\ 50288\ldots$$

The convention behind our decimal notation is that any number can be written as an infinite sum. Here it means that

$$\pi = 3 + \frac{1}{10} + \frac{4}{10^2} + \frac{1}{10^3} + \frac{5}{10^4} + \frac{9}{10^5} + \frac{2}{10^6} + \frac{6}{10^7} + \frac{5}{10^8} + \cdots$$

where the three dots $(\cdots)$ indicate that the sum continues forever, and the more terms we add, the closer we get to the actual value of π.

Since there are infinitely many terms to sum in an infinite series, we cannot just keep adding to determine the result. Instead, we begin by considering the sum of the first n terms of the sequence. We call this a partial sum.

In general to calculate $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \ldots + a_n + \ldots$, we consider the **partial sums**:

$$s_1 = a_1$$

$$s_2 = a_1 + a_2$$

$$s_3 = a_1 + a_2 + a_3$$

$$s_4 = a_1 + a_2 + a_3 + a_4$$

and, in general,

$$s_n = a_1 + a_2 + a_3 + \cdots + a_n = \sum_{i=1}^{n} a_i$$

These partial sums form a new sequence $\{s_n\}$, which may or may not have a limit.

If the limit of the partial sums exists, we call this limit the sum of the infinite series. In symbols:

$$\underbrace{\sum_{i=1}^{\infty} a_i}_{\textbf{infinite sum}} = \underbrace{\lim_{n\to\infty}\sum_{i=1}^{n} a_i}_{\substack{\textbf{limit of the}\\ \textbf{partial sums}}}$$

For example:

- $1+2+3+4+\ldots+n+\ldots$

- $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ldots+\frac{1}{2^{n-1}}+\ldots$

Is the sum of any finite number of terms in this series equal to 1?

Can we actually sum an infinite series term by term?

At the same time, we can determine the sum by defining it to be the limit of the sequence of partial sums as $n\to\infty$, in this case 1. Our knowledge of sequences and limits enables us to break away from the confines of finite sums.

$\boxed{2}$ **Definition** Given a series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$, let s_n denote its nth partial sum:

$$s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$$

If the sequence $\{s_n\}$ is convergent and $\lim_{n\to\infty} s_n = s$ exists as a real number, then the series $\Sigma\, a_n$ is called **convergent** and we write

$$a_1 + a_2 + \cdots + a_n + \cdots = s \qquad \text{or} \qquad \sum_{n=1}^{\infty} a_n = s$$

The number s is called the **sum** of the series. If the sequence $\{s_n\}$ is divergent, then the series is called **divergent**.

Furthermore, once we have convergent series, we can use the following theorem to combine and/or decompose series.

Warning: These rules only work for convergent series.

$\boxed{8}$ **Theorem** If $\Sigma\, a_n$ and $\Sigma\, b_n$ are convergent series, then so are the series $\Sigma\, ca_n$ (where c is a constant), $\Sigma\,(a_n + b_n)$, and $\Sigma\,(a_n - b_n)$, and

(i) $\displaystyle\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$

(ii) $\displaystyle\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$

(iii) $\displaystyle\sum_{n=1}^{\infty} (a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$

One challenge in finding the sum of a series is to find an expression the nth partial sum.

❖ Geometric Series

Geometric series are series of the form

$$a + ar + ar^2 + ar^3 + ... + ar^{n-1} + ... = \sum_{n=1}^{\infty} ar^{n-1}$$

In which a and r are fixed numbers and $a \neq 0$. The series can also be written as $\sum_{n=0}^{\infty} ar^n$. The **common ratio** r can be positive or negative.

<u>Derive the sum of the infinite geometric series</u>:

4 The geometric series

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots$$

is convergent if $|r| < 1$ and its sum is

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \qquad |r| < 1$$

If $|r| \geqslant 1$, the geometric series is divergent.

Ex1: Find the sum of the geometric series with $a = \frac{1}{9}$ and $r = \frac{1}{3}$.

Ex2: Does $\sum_{n=0}^{\infty} \frac{(-1)^n 5}{4^n}$ converge? If so find the sum.

Ex3: Express $5.\overline{23}$ as the ratio of two integers.

Ex4: Find the sum of $\sum_{n=1}^{\infty} \frac{3^{n-1}-1}{6^{n-1}}$.

Pro-tip: In this topic you should begin a list of famous series, tests, and methods. So far, we have one entry on THE LIST: (1.) The geometric series converges when $|r|<1$.

The Harmonic Series: Its name derives from the concept of overtones, or harmonics in music: the wavelengths of the overtones of a vibrating string are 1/2, 1/3, 1/4, … etc., of the string's fundamental wavelength. Every term of the series after the first is the harmonic mean of the neighboring terms.[1]

[1] The harmonic mean has formula $H = \frac{n}{\sum_{i=1}^{n} x_i^{-1}}$

The divergence of the harmonic series (which we just showed) was first proven in the 14th century by Nicole Oresme, but this achievement fell into obscurity. Proofs were given in the 17th century by Pietro Mengoli, Johann Bernoulli, and Jacob Bernoulli.

Historically, harmonic sequences have had a certain popularity with architects. This was so particularly in the Baroque period, when architects used them to establish the proportions of floor plans, of elevations, and to establish harmonic relationships between both interior and exterior architectural details of churches and palaces.

We are now ready to update THE LIST: (1.) The geometric series converges when $|r|<1$. (2.) The harmonic series diverges.

Telescoping Series: With geometric series, we carried out the entire evaluation process by finding a formula for the sequence of partial sums and evaluating the limit of the sequence. Not many infinite series can be subjected to this sort of analysis. With another class of series, called telescoping series, it can be done.

Ex5: Evaluate $\sum_{n=1}^{\infty} \frac{2}{n^2+4n+3}$

We end the section with one more addition to THE LIST: (1.) The geometric series converges when $|r|<1$. (2.) The harmonic series diverges. (3.) Telescoping series