Name: Key

Assessment 7

Math& 264: Multivariable Calculus

Instructions: Please carefully complete these questions by hand. Be sure to show all work including sketching relevant graphs and providing exact answers.

Should you choose to work these on scratch paper, please do not put more than one question on a page. Additional sheets of paper are acceptable.

Upload your solutions to Gradescope by 9 am on Tuesday (5/26). During your presentation time, you will be asked to explain your thought process and reasoning on one randomly assigned question. Late solutions are available thru 2 pm with a 2 point penalty (smaller penalties if you can document that this is a revision and only minor changes).

Please make sure to sign up for your presentation slot. If you are unavailable for any of the times available, please send me a note in Slack and we will find a time that works for you.

https://docs.google.com/spreadsheets/d/1IGUR3J5qnXkbLUhWWeoMjbxYb1peN9lt5gZYj7wAtCo/edit?usp=sharing

It is Memorial Day weekend. In honor of this day, you may all celebrate by foregoing (skipping) one exercise. Furthermore, if you were/are a member of the armed forces and/or had a direct relative die while serving in the military, you may forego a second exercise. (Given we are at Highline, military service for other nations counts.) Please provide a brief description/explanation so I may join with you in honoring the service of those we remember.

(1.1) Given the force field $\vec{F} = \langle y, -x \rangle$, find the work required to move an object on the path that follows the line segment from (1,6) to (0,0) followed by the line segment from (0,0) to (0,8).

$W_1 = \int_{C_1} \vec{F} \cdot d\vec{r}$

$\vec{r}_1(t) = (1-t)\langle 1, 6\rangle$ on $0 \le t \le 1$

$= \langle 1-t, 6-6t \rangle$

$d\vec{r} = \langle -1, -6 \rangle dt$

$\Rightarrow W_1 = \int_0^1 \langle 6-6t, t-1 \rangle \cdot \langle -1, -6 \rangle dt$

$= \int_0^1 (-6+6t-6t+6)dt$

$= 0$

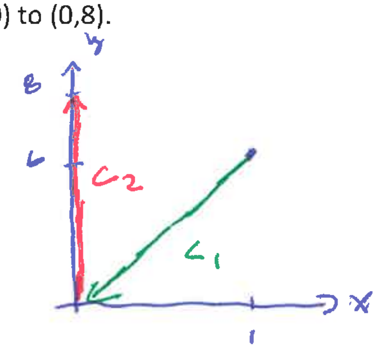
$W_2 = \int_{C_2} \vec{F} \cdot d\vec{r}$

$\vec{r}_2(t) = \langle 0, 8t \rangle$ on $0 \le t \le 1$

$d\vec{r} = \langle 0, 8 \rangle dt$

$\Rightarrow W_2 = \int_0^1 \langle 8t, 0 \rangle \cdot \langle 0, 8 \rangle dt$

$= 0$



The total work is 0.

(1.2) Given the vector field $\vec{F} = \frac{\langle x, y\rangle}{(x^2+y^2)^{3/2}}$ on the curve $\vec{r}(t) = \langle 3t^2, 7t^2\rangle$ for $1 \le t \le 2$, evaluate and interpret $\int_C \vec{F}\cdot\vec{T}\,ds$. ← work

$$\int_C \vec{F}\cdot\vec{T}\,ds = \int_C \vec{F}\cdot d\vec{r}$$

$$= \int_1^2 \frac{\langle 3t^2, 7t^2\rangle}{\sqrt{9t^4+49t^4}^3}\cdot\langle 6t, 14t\rangle\,dt$$

$$= \frac{1}{\sqrt{58}^3}\int_1^2 \frac{116t^3}{t^6}\,dt$$

$$= \frac{2}{\sqrt{58}}\int_1^2 t^{-3}\,dt$$

$$\to \frac{2}{\sqrt{58}}\left[-\frac{1}{2}\cdot t^{-2}\right]_1^2$$

$$= \frac{-1}{\sqrt{58}}\left(\frac{1}{4}-1\right)$$

$$= \frac{3}{4\sqrt{58}}$$

This is the work to travel along $\vec{r}(t)$ from $(3,7)$ to $(12,28)$.

(1.3) For what values of a and d does the vector field $F = \langle ax, dy\rangle$ have zero flux across the unit circle centered at the origin and oriented counterclockwise?

$$\text{Flux} = \int_C f\,dy - g\,dx$$

$\vec{r}(t) = \langle \cos t, \sin t\rangle$ on $0 \le t \le 2\pi$

$d\vec{r} = \langle -\sin t, \cos t\rangle\,dt$

$$= \int_0^{2\pi}[a\cos t(\cos t) - d\sin t(-\sin t)]\,dt$$

$$= \int_0^{2\pi}\left[\frac{a}{2}(1+\cos 2t) + \frac{d}{2}(1-\cos 2t)\right]dt$$

$$= \left[\frac{a}{2}\left[t+\frac{1}{2}\sin(2t)\right] + \frac{d}{2}\left[t-\frac{1}{2}\sin 2t\right]\right]_0^{2\pi}$$

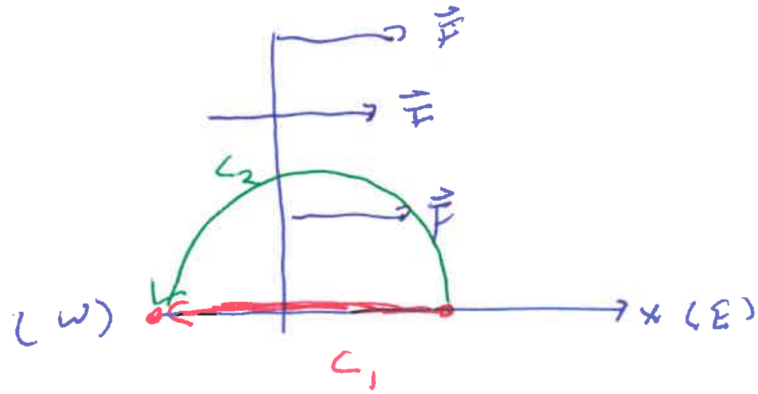
$$= \pi(a+d)$$

The flux is zero when $a+d=0$

(1.4) An airplane flies in the *xz*-plane, where *x* increases in the eastward direction and $z \geq 0$ represents vertical distance above the ground. A wind blows horizontally out of the west producing a force $\vec{F} = \langle 200, 0 \rangle$. On which of the two paths between (110,0) and (-110,0) is the most work done: $\vec{r}_1(t) = \langle -t, 0 \rangle$ or $\vec{r}_2(t) = \langle 110\cos t, 110\sin t \rangle$. Justify your answer mathematically.

$$W_1 = \int_{C_1} \vec{F} \cdot d\vec{r}$$

$$= \int_{-110}^{110} \langle 200, 0 \rangle \cdot \langle -1, 0 \rangle \, dt$$



$$= -200(220)$$

$$= -44000$$

$$W_2 = \int_{C_2} \vec{F} \cdot dr$$

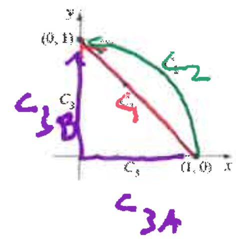
$$= \int_0^{\pi} \langle 200, 0 \rangle \cdot \langle -110\sin t, 110\cos t \rangle \, dt$$

$$= \int_0^{\pi} -22000 \sin t \, dt$$

$$= -22000(2)$$

The same amount of work is done.

(1.5) Consider the rotation field $\vec{F} = \langle -y, x \rangle$ and the three paths shown in the figure. Compute the work done on each of the three paths. Does it appear that the line integral $\int_C \vec{F} \cdot \vec{T}\, ds$ is independent of the path from (1,0) to (0,1)?



C_1: $\vec{r}_1(t) = (1-t)\langle 1,0\rangle + t\langle 0,1\rangle$ on $0 \le t \le 1$

$= \langle 1-t, t\rangle$

$d\vec{r} = \langle -1, 1\rangle$

$W_1 = \int_0^1 \langle -t, 1-t\rangle \cdot \langle -1, 1\rangle\, dt$

$= \int_0^1 t + 1 - t\, dt = \boxed{1}$

C_2: $\vec{r}_2(t) = \langle \cos t, \sin t\rangle$ on $0 \le t \le \frac{\pi}{2}$

$W_2 = \int_0^{\frac{\pi}{2}} \langle -\sin t, \cos t\rangle \cdot \langle -\sin t, \cos t\rangle\, dt = \boxed{\frac{\pi}{2}}$

C_{3A}: $\vec{r}_{3A}(t) = \langle 1-t, 0\rangle$ on $0 \le t \le 1$

$W_{3A} = \int_0^1 \langle 0, 1-t\rangle \cdot \langle -1, 0\rangle\, dt = 0$

C_{3B}: $\vec{r}_{3B} = \langle 0, t\rangle$ on $0 \le t \le 1$ $\boxed{0}$

$W_{3B} = \int_0^1 \langle -t, 0\rangle \langle 0, 1\rangle\, dt = 0$

work is <u>not</u> independent of path.

$\langle y, x \rangle$

(1.6) Evaluate the line integral $\int_C \nabla\varphi \cdot d\bar{r}$ for the potential function $\varphi(x, y) = xy$ in two ways: directly along the path $\bar{r}(t) = \langle \sin t, \cos t \rangle$ for $0 \le t \le \frac{\pi}{3}$ and also indirectly using the Fundamental Theorem for Line Integrals.

Directly: $\int_0^{\pi/3} \langle \cos t, \sin t \rangle \cdot \langle \cos t, -\sin t \rangle \, dt$

$= \int_0^{\pi/3} \cos 2t \, dt$

$= \frac{1}{2}\left[\sin 2t\right]_0^{\pi/3}$

$= \frac{1}{2} \cdot \frac{\sqrt{3}}{2}$

FToLI: $\int_C \nabla\varphi \cdot d\vec{r} = \underbrace{\varphi(B)}_{\frac{\sqrt{3}}{2}\cdot\frac{1}{2}} - \underbrace{\varphi(A)}_{0}$

Both methods yield the same result: $\frac{\sqrt{3}}{4}$

(1.7) Verify that the Fundamental Theorem for Line Integrals can be used to evaluate the line integral $\int_C \nabla\underbrace{\left(e^{-y}\sin x\right)}_{\varphi} \cdot d\bar{r}$ where *C* is the line from (0,0) to $(\pi, \ln 7)$. Then evaluate the integral.

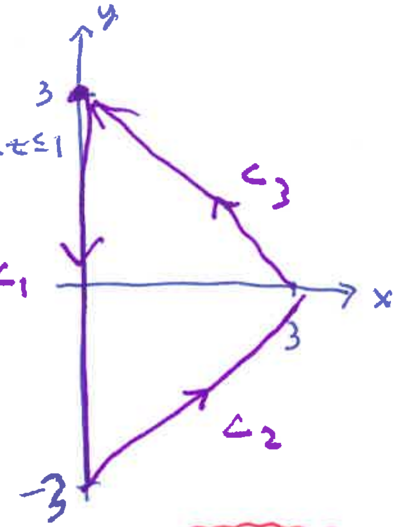
$\langle e^{-y}\cos x, -e^{-y}\sin x \rangle$

verify: $\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} = -e^{-y}\cos x - (-e^{-y}\cos x) = 0 \checkmark$

work $= \underbrace{\varphi(\pi, \ln 7)}_{\frac{1}{e^7}(0)} - \underbrace{\varphi(0,0)}_{0} = 0$

(1.8) Let $\vec{F} = \langle x, y \rangle$ and C be the triangle with vertices (0,3), (0,-3), and (3,0) oriented CCW. Evaluate $\int_C \vec{F} \cdot d\vec{r}$ directly, using the Fundamental Theorem for Line Integrals, and using Green's Theorem.

Directly:



C_1: $\vec{r}_1(t) = \langle 0, 3-6t \rangle$ on $0 \le t \le 1$

$$W_1 = \int_0^1 \langle 0, 3-6t \rangle \cdot \langle 0, -6 \rangle \, dt$$
$$= \int_0^1 (-18 + 36t)\, dt$$
$$= \left[-18t + 18t^2\right]_0^1$$
$$= 0$$

C_2: $\vec{r}_2(t) = \langle 3t, -3+3t \rangle$ on $0 \le t \le 1$

$$W_2 = \int_0^1 \langle 3t, -3+3t \rangle \cdot \langle 3, 3 \rangle \, dt$$
$$= \int_0^1 (9t - 9 + 9t)\, dt$$
$$= \left[-9t + 9t^2\right]_0^1$$
$$= 0$$

C_3: $\vec{r}_3(t) = \langle 3-3t, 3t \rangle$ on $0 \le t \le 1$

$$W_3 = \int_0^1 \langle 3-3t, 3t \rangle \cdot \langle -3, 3 \rangle \, dt$$
$$= \int_0^1 (-9 + 9t + 9t)\, dt$$
$$= 0$$

calculated directly, the work is $\boxed{0}$

FToLI

$$\frac{\partial g}{\partial y} - \frac{\partial f}{\partial x} = 1 - 1 = 0$$

since the field is conservative

$$\oint_C \vec{F} \cdot d\vec{r} = 0$$

Green's Thm

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_R \frac{\partial g}{\partial y} - \frac{\partial f}{\partial x}\, dA$$
$$= \iint_R 0 \, dA = 0$$

(1.9) Find the work required to move an object in the force field $\vec{F} = e^{x+y}\langle 1,1,z\rangle$ along the straight line from A(0,0,0) to B(-1,2,-1).

$= \langle e^{x+y}, e^{x+y}, ze^{x+y}\rangle$

Is the field conservative?

$f_y = e^{x+y} = g_x \,;\; f_z = e^{x+y} = h_x \,;\; g_z = 0 \neq ze^{x+y} = h_y$

$\uparrow$

Not conservative

$\vec{r}(t) = (1-t)\langle -1, 2, -1\rangle = \langle t-1, 2-2t, t-1\rangle$ on $0 \le t \le 1$

This path is from B to A.

$$W = \int_0^1 e^{1-t}\langle 1, 1, t-1\rangle \cdot \langle 1, -2, 1\rangle \, dt$$

$$= \int_0^1 e^{1-t} \underbrace{(1 - 2 + t - 1)}_{t-2} \, dt$$

$$= \int_0^1 (t-2)e^{1-t} \, dt$$

Let $u = 1-t$
$du = -dt$

$$= -\int_1^0 ((1-u)-2)e^u \, du$$

$$= -\int_0^1 (1+u)e^u \, du$$

$$= -\left[\int_0^1 e^u \, du + \underbrace{\int_0^1 ue^u \, du}_{ue^u - \int e^u du}\right]$$

$$= -\left[ue^u\right]_0^1$$

$$= -e$$

We want the work from A to B which is e.

(1.10) The gravitational force between two point masses M and m is $\vec{F} = GMm = \dfrac{GMm\langle x, y, z\rangle}{\left(x^2+y^2+z^2\right)^{3/2}}$

where G is the gravitational constant. Does the work depend on the path? Explain.

To do this, verify that the force field is conservative on any region excluding the origin, find the potential function, and then find the work required to move an object with mass m from A to B where A is a distance of r_1 from M and B is a distance of r_2 from M.

Is $\vec{F}$ conservative?

$$f_y = \frac{-3GMmxy}{(x^2+y^2+z^2)^{5/2}} = g_x$$

$$f_z = \frac{-3GMmxz}{(x^2+y^2+z^2)^{5/2}} = h_x$$

$$g_z = \frac{-3GMmyz}{(x^2+y^2+z^2)^{5/2}} = h_y$$

yes, $\vec{F}$ is conservative, ∴ The work is independent of the path,

$$② \quad f = \frac{GMmx}{(x^2+y^2+z^2)^{3/2}}$$

$$\Big\downarrow \int$$

$$① \quad \varphi(x,y,z) = \frac{-GMm}{\sqrt{x^2+y^2+z^2}} + c(y,z)$$

$$\Big\downarrow \frac{\partial}{\partial y}$$

$$② \quad \frac{+GMmy}{(x^2+y^2+z^2)^{3/2}} + c_y(y,z) = g \Rightarrow c_y = 0$$

scratch

$$\int \frac{GMmx}{(x^2+y^2+z^2)^{3/2}}\,dx$$

Let $u = x^2+y^2+z^2$

$$\frac{du}{2} = x\,dx$$

$$= \frac{1}{2} GMm \int \frac{du}{u^{3/2}}$$

$$= -GMm\,u^{-\frac{1}{2}} + c(y,z)$$

$$= -\frac{GMm}{\sqrt{x^2+y^2+z^2}} + c(y,z)$$

$$\downarrow \int c_y\,dy = D(z)$$

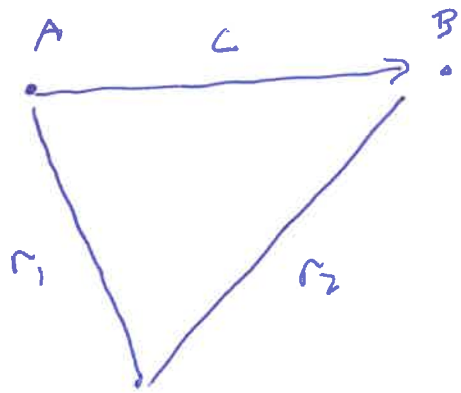
③ $$\varphi(x,y,z) = \frac{-GMm}{\sqrt{x^2+y^2+z^2}} + D(z)$$

$$\downarrow \frac{d}{dz}$$

④ $$\frac{GMmz}{(x^2+y^2+z^2)^{3/2}} + D'(z) = h \Rightarrow D'(z) = 0$$

$$\downarrow \int D'(z)\,dz$$

⑤ $$\varphi(x,y,z) = \frac{-GMm}{\sqrt{x^2+y^2+z^2}}$$



$$\text{work} = \int_C \vec{F} \cdot d\vec{r}$$

$$= \int_C \nabla \varphi \cdot d\vec{r}$$

$$= \varphi(B) - \varphi(A)$$

$$= -\frac{GMm}{r_2} - \frac{-GMm}{r_1}$$

$$= GMm\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

Notice that the work does not depend upon the path.