Name: Key

Assessment 6

Math& 264: Multivariable Calculus

Instructions: Please carefully complete these questions by hand. Be sure to show all work including sketching relevant graphs and providing exact answers.

Should you choose to work these on scratch paper, please do not put more than one question on a page. Additional sheets of paper are acceptable.

Upload your solutions to Gradescope by 9 am on Monday (5/18). During your presentation time, you will be asked to explain your thought process and reasoning on one randomly assigned question. Late solutions are available thru 2 pm with a 2 point penalty (smaller penalties if you can document that this is a revision and only minor changes).

Please make sure to sign up for your presentation slot. If you are unavailable for any of the times available, please send me a note in Slack and we will find a time that works for you.

https://docs.google.com/spreadsheets/d/1IGUR3J5qnXkbLUhWWeoMjbxYb1peN9lt5gZYj7wAtCo/edit?usp=sharing

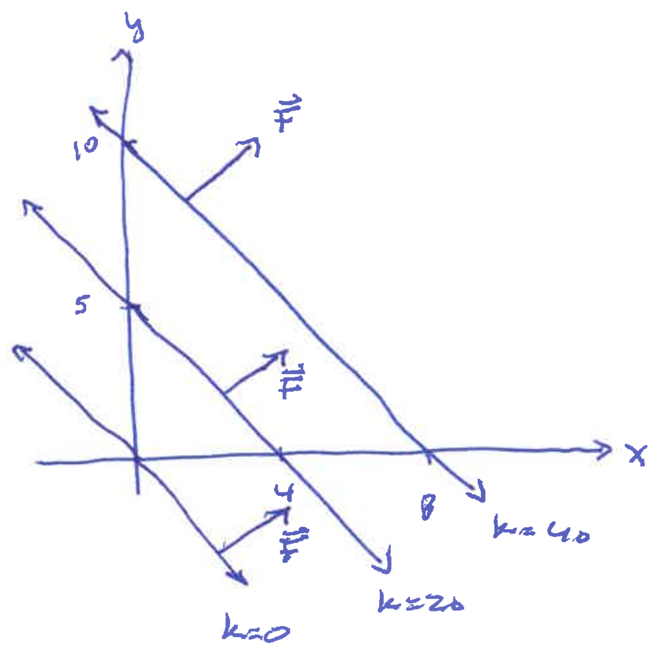
(1.1) Find the gradient field $\vec{F} = \nabla\varphi$ for the potential function $\varphi = 5x + 4y$ for $|x| \le 10$ and $|y| \le 10$. Sketch a few level curves of φ and a few vectors of $\vec{F}$.

$\nabla\varphi = \langle 5, 4 \rangle$

Level curves

$k = 5x + 4y$

k	curve
0	$0 = 5x + 4y$
20	$20 = 5x + 4y$
40	$40 = 5x + 4y$



(1.2) Consider the potential function $\varphi(x,y)=e^{x-y}$. Pick a point and show that the level curves and gradient are orthogonal.

$\nabla \varphi = \langle e^{x-y}, -e^{x-y} \rangle$ at the point (a,b)

The level curve thru (a,b) is $\underbrace{e^{a-b}}_{\text{a number}} = e^{x-y}$

$$\Rightarrow e^{a-b} = \frac{e^x}{e^y} \Rightarrow e^y \cdot e^{a-b} = e^x$$

$$\Rightarrow \ln(e^y) + \ln(e^{a-b}) = \ln(e^x)$$

$$\Rightarrow y = x + (b-a)$$

The vector $\langle 1,1 \rangle$ is parallel to the level curves.

$\langle 1,1 \rangle \cdot \langle e^{x-y}, -e^{x-y} \rangle = 0$ $\therefore$ the gradient and level curves are orthogonal.

(1.3) Determine all values of x for which the series $\sum_{k=1}^{\infty} \frac{8x^{2k}}{3k^2}$ converges. Show work and justify results.

ratio test

$$\lim_{k\to\infty} \left| \frac{8x^{2(k+1)}}{3(k+1)^2} \cdot \frac{3k^2}{8x^{2k}} \right| = \lim_{k\to\infty} \left| \frac{x^2 k^2}{(k+1)^2} \right| = |x^2| < 1$$

The series converges when $-1 < x < 1$.

check end pts

$x = 1$: $\sum_{k=1}^{\infty} \frac{8}{3k^2}$ convergent p-series

$x = -1$: $\sum_{k=1}^{\infty} \frac{8}{3k^2}$ convergent p-series.

So the interval of convergence is $-1 \le x \le 1$.

(1.4) Determine if the series $\sum_{N=1}^{\infty}(-1)^n\left(1-\frac{6}{n}\right)^{n^2}$ converges or diverges. If the series converges, does it converge conditionally or absolutely? Show work and justify results.

root test.

$$\lim_{N\to\infty}\sqrt[N]{\left|(-1)^N\left(1-\frac{6}{N}\right)^{N^2}\right|}=\lim_{N\to\infty}\left(1-\frac{6}{N}\right)^N$$

$$= e^{\lim_{N\to\infty} N\cdot\ln\left(1-\frac{6}{N}\right)}$$

$$= e^{\lim_{x\to\infty}\frac{\ln\left(1-\frac{6}{x}\right)}{\frac{1}{x}}}$$

$$\overset{(H)}{=} e^{\lim_{x\to\infty}\frac{\frac{1}{1-\frac{6}{x}}\cdot\frac{6}{x^2}}{-\frac{1}{x^2}}}$$

$$= e^{\lim_{x\to\infty}\frac{-6}{1-6/x}}$$

$$= e^{-6} < 1$$

∴ The series converges absolutely by the root test.

(1.5) Determine if the series $\sum_{k=1}^{\infty}\ln\left(\frac{k+8}{k+7}\right)$ converges or diverges. If the series converges, does it converge conditionally or absolutely? Show work and justify results.

$$\sum_{k=1}^{\infty}\ln\left(\frac{k+8}{k+7}\right)=\sum_{k=1}^{\infty}\left[\ln(k+8)-\ln(k+7)\right]$$

$$=(\cancel{\ln 9}-\ln 8)+(\cancel{\ln 10}-\cancel{\ln 9})+(\ln 11-\cancel{\ln 10})+\ldots$$

$$= -\ln 8+\lim_{k\to\infty}\ln(k+8)$$

$$= \infty$$

∴ The series is a divergent telescoping series.

$\boxed{\text{ratio test}}$

$$\lim_{k\to\infty} \frac{14^{k+1}(k+1)!}{(k+1)^{k+1}} \cdot \frac{k^k}{14^k k!} = \lim_{k\to\infty} 14\left(\frac{k}{1+k}\right)^k = 14^{\lim_{k\to\infty}}(1+\tfrac{1}{k})^{-k}$$

(1.6) Determine if the series $\sum_{k=1}^{\infty}\left(\frac{14^k k!}{k^k}\right)$ converges or diverges. If the series converges, does it converge conditionally or absolutely? Show work and justify results.

$$= 14\, e^{\lim_{x\to\infty} -x\ln(1+\frac{1}{x})}$$

$$= 14\, e^{\lim_{x\to\infty} \frac{-\ln(1+\frac{1}{x})}{\frac{1}{x}}}$$

$$\overset{H}{=} 14\, e^{\lim_{x\to\infty} \frac{\frac{-1}{1+\frac{1}{x}}\cdot -\frac{1}{x^2}}{-\frac{1}{x^2}}}$$

$$= 14e^{-1} > 1$$

$\therefore$ the series diverges by the ratio test.

(1.7) Determine if the series $\sum_{n=1}^{\infty}\frac{n!}{(2n+6)!}$ converges or diverges. If the series converges, does it converge conditionally or absolutely? Show work and justify results.

$\boxed{\text{ratio test}}$

$$\lim_{n\to\infty} \frac{(n+1)!}{(2(n+1)+6)!} \cdot \frac{(2n+6)!}{n!} = \lim_{n\to\infty} \frac{(n+1)}{(2n+8)(2n+7)} = 0 < 1$$

$\therefore$ The series converges absolutely by the ratio test.

(1.8) Determine if the series $\sum_{k=1}^{\infty}\frac{8^k}{13^k - 8^k}$ converges or diverges. If the series converges, does it converge conditionally or absolutely? Show work and justify results.

$\boxed{\text{limit comparison test}}$

$$\lim_{k\to\infty} \frac{\frac{8^k}{13^k}}{\frac{8^k}{13^k-8^k}} = \lim_{k\to\infty} \frac{\cancel{8^k}}{13^k} \cdot \frac{13^k - 8^k}{\cancel{8^k}} = \lim_{k\to\infty} 1 - \left(\frac{8}{13}\right)^k = 1$$

and $\sum_{k=1}^{\infty}\frac{8^k}{13^k}$ is a convergent geometric series.

The terms are all positive so $\sum_{k=1}^{\infty}\frac{8^k}{13^k-8^k}$ is absolutely convergent by the limit comparison test.

(1.9) Determine if the series $\sum_{k=3}^{\infty} \frac{1}{k^{2/9} \ln k}$ converges or diverges. If the series converges, does it converge conditionally or absolutely? Show work and justify results.

$\boxed{\text{comparison \& integral tests}}$

$$\sum_{k=3}^{\infty} \frac{1}{k^{2/9} \ln k} > \sum_{k=3}^{\infty} \frac{1}{k \ln k}$$

$$\int_3^{\infty} \frac{dx}{x \ln x} = \lim_{b \to \infty} \Big[\ln(\ln x) \Big]_3^b = \lim_{b \to \infty} \big(\ln(\ln b) - \ln(\ln 3) \big) = \infty$$

$\therefore \sum_{k=3}^{\infty} \frac{1}{k \ln k}$ diverges by the integral test

and $\sum_{k=3}^{\infty} \frac{1}{k^{2/9} \ln k}$ diverges by the comparison test.

(1.10) Determine if the series $\sum_{k=1}^{\infty} \left(1 - \frac{a}{k}\right)^{4k}, a \in \mathbb{R}$ converges or diverges. If the series converges, does it converge conditionally or absolutely? Show work and justify results.

Test for Divergence.

$$\lim_{k \to \infty} \left(1 - \frac{a}{k}\right)^{4k} = e^{\lim_{k \to \infty} 4k \ln\left(1 - \frac{a}{k}\right)}$$

$$= e^{\lim_{x \to \infty} \frac{4 \ln\left(1 - \frac{a}{x}\right)}{\frac{1}{x}}}$$

$$\overset{(H)}{=} e^{\lim_{x \to \infty} \frac{\frac{4}{1 - \frac{a}{x}} \cdot \frac{a}{x^2}}{-\frac{1}{x^2}}}$$

$$= e^{\lim_{x \to \infty} \frac{-4a}{1 - \frac{a}{x}}}$$

$$= e^{-4a}$$

$$\neq 0$$

$\therefore$ The series diverges by the test for divergence.