Name: Key.

Assessment 5

Math& 264: Multivariable Calculus

Instructions: Please carefully complete these questions by hand. Be sure to show all work including sketching relevant graphs and providing exact answers.

Should you choose to work these on scratch paper, please do not put more than one question on a page. Additional sheets of paper are acceptable.

Upload your solutions to Gradescope by 9 am on Monday (5/11). During your presentation time, you will be asked to explain your thought process and reasoning on one randomly assigned question. Late solutions are available thru noon with a 2 point penalty (smaller penalties if you can document that this is a revision and only minor changes).

Please make sure to sign up for your presentation slot. If you are unavailable for any of the times available, please send me a note in Slack and we will find a time that works for you.

https://docs.google.com/spreadsheets/d/1lGUR3J5qnXkbLUhWWeoMjbxYb1peN9lt5gZYj7wAtCo/edit?usp=sharing

It is Mother's Day weekend. Choose one question to skip and just put "Mother's Day" as your answer. If you happen to be a mother, choose two questions to skip.

(1.1) Does the series $\sum_{k=1}^{\infty} \frac{k}{e^k}$ converge or diverge? Justify your answer and be sure to state a clear and logical conclusion.

$$\int_1^\infty xe^{-x}\,dx = \left[-xe^{-x} + \int e^{-x}\right]_1^\infty$$

$u = x \qquad dv = e^{-x}dx$

$du = dx \qquad v = -e^{-x}$

$$= \left[-xe^{-x} - e^{-x}\right]_1^\infty$$

$$= \lim_{t\to\infty}\left[\left(\frac{-t}{e^{+t}} - \cancel{e^{-t}}^{\,0}\right) - \left(-\frac{1}{e} - \frac{1}{e}\right)\right]$$

$$= \frac{2}{e} - \lim_{t\to\infty}\frac{t}{e^t}$$

$$\overset{H}{=} \frac{2}{e} - \lim_{t\to\infty}\cancel{\frac{1}{e^t}}^{\,0}$$

$$= \frac{2}{e}$$

Since $\int_1^\infty xe^{-x}dx$ converges, $\sum_{k=1}^{\infty} ke^{-k}$ converges by the integral test.

(1.2) Consider the series $\sum_{k=1}^{\infty} \frac{1}{3^k}$. Find how many terms are needed to ensure that the remainder is less than 10^{-3}. Find an interval in which the value of the series must lie if you approximate it using the first 10 terms of the series.

$$R_N < \int_N^{\infty} \left(\frac{1}{3}\right)^x dx \leq 10^{-3}$$

$$\Rightarrow \int_N^{\infty} e^{x \ln\left(\frac{1}{3}\right)} dx \leq 10^{-3}$$

$$\Rightarrow \lim_{t \to \infty} \left[\frac{1}{\ln\left(\frac{1}{3}\right)} e^{x \ln\left(\frac{1}{3}\right)}\right]_N^t \leq 10^{-3}$$

$$\Rightarrow \frac{1}{\ln\left(\frac{1}{3}\right)} \lim_{t \to \infty} \left(\cancel{e^{t \ln\left(\frac{1}{3}\right)}} - e^{N \ln\left(\frac{1}{3}\right)}\right) \leq 10^{-3}$$

$$\Rightarrow \frac{-1}{\ln\left(\frac{1}{3}\right)} \cdot \left(\frac{1}{3}\right)^N \leq 10^{-3}$$

$\Rightarrow \left(\frac{1}{3}\right)^N \leq -\ln\left(\frac{1}{3}\right) 10^{-3}$

$\Rightarrow N \ln\left(\frac{1}{3}\right) \leq \ln\left(-\ln\left(\frac{1}{3}\right)10^{-3}\right)$

$\Rightarrow N \geq \dfrac{\ln\left(-\ln\left(\frac{1}{3}\right)10^{-3}\right)}{\ln\left(\frac{1}{3}\right)}$

$\Rightarrow N \geq 6.2$

we need 7 terms.

Also $5.14 \times 10^{-6} < R_{10} < 1.54 \times 10^{-5}$

see attached

(1.3) Does the series $\sum_{k=3}^{\infty} \frac{4}{k\sqrt{\ln k}}$ converge or diverge? Justify your answer and be sure to state a clear and logical conclusion.

$$\int_3^{\infty} \frac{4}{x\sqrt{\ln x}} dx = \int_{\ln 3}^{\infty} \frac{4}{\sqrt{u}} du$$

which diverges by the p-test for improper integrals.

Let $u = \ln x$

$du = \dfrac{dx}{x}$

$\therefore$ since $\int_3^{\infty} \frac{4\,dx}{x\sqrt{\ln x}}$ diverges, $\sum_{k=3}^{\infty} \frac{4}{k\sqrt{\ln k}}$ diverges by the integral test.

(1.4) Does the series $\sum_{k=1}^{\infty} \frac{k^2+2k+1}{3k^2+1}$ converge or diverge? Justify your answer and be sure to state a clear and logical conclusion.

$$\lim_{k\to\infty} \frac{k^2+2k+1}{3k^2+1} = \frac{1}{3}$$

$\therefore$ The series diverges by the test for divergence.

(1.5) Does the series $\sum_{k=1}^{\infty} \frac{k^8}{k^{11}-3}$ converge or diverge? Justify your answer and be sure to state a clear and logical conclusion.

$$\lim_{k\to\infty} \frac{\frac{k^8}{k^{11}-3}}{\frac{1}{k^3}} = \lim_{k\to\infty} \frac{k^{11}}{k^{11}-3} = 1$$

$\therefore$ since $\sum_{k=1}^{\infty} \frac{1}{k^3}$ is a convergent p-series,

$\sum_{k=1}^{\infty} \frac{k^8}{k^{11}-3}$ converges by the limit comparison test.

(1.6) Does the series $\frac{1}{2^2}+\frac{2}{3^2}+\frac{3}{4^2}+\ldots$ converge or diverge? Justify your answer and be sure to state a clear and logical conclusion.

$$\frac{1}{2^2}+\frac{2}{3^2}+\frac{3}{4^2}+\ldots = \sum_{k=1}^{\infty} \frac{k}{(k+1)^2}$$

$$\lim_{k\to\infty} \frac{\frac{k}{(k+1)^2}}{\frac{1}{k}} = \lim_{k\to\infty} \frac{k^2}{(k+1)^2} = 1$$

$\therefore$ since $\sum_{k=1}^{\infty} \frac{1}{k}$ is the divergent harmonic series, the given series diverges by the limit comparison test.

(1.7) Does the series $\sum_{k=3}^{\infty} \frac{1}{\ln k}$ converge or diverge? Justify your answer and be sure to state a clear and logical conclusion.

$$\lim_{k \to \infty} \frac{\frac{1}{\ln k}}{\frac{1}{k}} = \lim_{k \to \infty} \frac{k}{\ln k} \overset{H}{=} \lim_{k \to \infty} \frac{1}{1/k} = \infty$$

$\therefore$ since $\sum_{k=3}^{\infty} \frac{1}{k}$ diverges (harmonic series)

we know $\sum_{k=3}^{\infty} \frac{1}{\ln k}$ diverges by the

limit comparison test.

(1.8) How many terms of the series $\frac{1}{e} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}$ must be summed to be sure that the remainder is less than 10^{-4} in magnitude?

$$a_k = \frac{1}{k!} \leq 10^{-4} \Rightarrow 10000 \leq k!$$

$$\Rightarrow k \geq 8 \quad (8 \text{ terms})$$

so $\underbrace{\sum_{k=0}^{7} \frac{(-1)^k}{k!}}_{103/280 \approx 0.36786}$ is w/in 10^{-4} of $\frac{1}{e}$

(1.9) Estimate the value of the series $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^5}$ with an absolute value less than 10^{-3} error w/

$$a_{k+1} = \frac{1}{(k+1)^5} \leq 10^{-3}$$

$$\Rightarrow 1000 \leq (k+1)^5$$

$$\Rightarrow \underbrace{\sqrt[5]{1000} - 1}_{\approx 2.98} \leq k$$

choose $k = 3$ (3 terms)

$$\Rightarrow \sum_{k=1}^{\infty} \frac{(-1)^k}{k^5} = \sum_{k=1}^{3} \frac{(-1)^k}{k^5} \pm 10^{-3}$$

$$= -0.9729 \pm 10^{-3}$$

(1.10) Determine whether the series $\sum_{k=1}^{\infty} \frac{\cos k}{k^3}$ converges absolutely, converges conditionally, or diverges.

$$\sum_{k=1}^{\infty} \left| \frac{\cos k}{k^3} \right| \leq \sum_{k=1}^{\infty} \frac{1}{k^3}$$ which is a convergent p-series.

$$\therefore \sum_{k=1}^{\infty} \left| \frac{\cos(k)}{k^3} \right|$$ converges by comparison

and $$\sum_{k=1}^{\infty} \frac{\cos(k)}{k^3}$$ converges absolutely.

(1.2 cont)

$$\int_{11}^{\infty} \left(\frac{1}{3}\right)^x dx < R_{10} < \int_{10}^{\infty} \left(\frac{1}{3}\right)^x dx$$

$$\Rightarrow \underbrace{\frac{-1}{\ln\left(\frac{1}{3}\right)} \cdot \left(\frac{1}{3}\right)^{11}}_{5.14 \times 10^{-6}} < R_{10} < \underbrace{\frac{1}{\ln 3} \left(\frac{1}{3}\right)^{10}}_{1.54 \times 10^{-5}}$$