Name: Key

Assessment 4

Math& 264: Multivariable Calculus

Instructions: Please carefully complete these questions by hand. Be sure to show all work including sketching relevant graphs and providing exact answers.

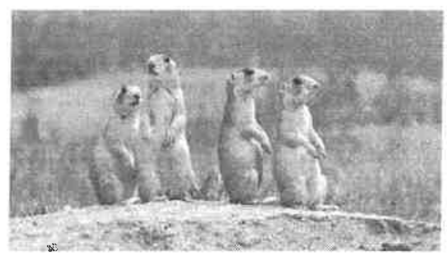
Should you choose to work these on scratch paper, please do not put more than one question on a page. Additional sheets of paper are acceptable.

Upload your solutions to Gradescope by 9 am on Monday (5/4). During your presentation time, you will be asked to explain your thought process and reasoning on one randomly assigned question.

Please make sure to sign up for your presentation slot. If you are unavailable for any of the times available, please send me a note in Slack and we will find a time that works for you.

https://docs.google.com/spreadsheets/d/1IGUR3J5qnXkbLUhWWeoMjbxYb1peN9lt5gZYj7wAtCo/edit?usp=sharing

(1.1) When a biologist begins a study, a colony of prairie dogs has a population of 250. Regular measurements reveal that each month the prairie dog population increases by 3%. Let p_n be the population (rounded to whole numbers) at the end of the nth month, where the initial population is $p_0 = 250$.



a.) Find an explicit formula for the sequence.

b.) Find a recurrence relation for the sequence.

c.) What is the limit of the sequence?

(a) $P_N = 250(1.03)^N$

(b) $P_0 = 250$

$P_{N+1} = 1.03 P_N$ for $N \geq 1$

(c) $\lim_{N \to \infty} P_N = \infty$

(1.2) A ball is thrown upward to a height of 20 meters. After each bounce, the ball rebounds to two-thirds of its previous height. Let h_n be the height after the nth bounce and let S_n be the total distance the ball has traveled at the end of the nth bounce.

a.) Find the first four terms of the sequence $\{S_n\}$.

b.) Determine a plausible value for the limit of $\{S_n\}$. $\frac{1}{2}S_0 = 20$

$h_0 = 20$

a) $h_1 = 20\left(\frac{2}{3}\right)$ $\qquad \frac{1}{2}S_1 = 20\left(\frac{2}{3}\right)$

$h_2 = 20\left(\frac{2}{3}\right)^2$ $\qquad \frac{1}{2}S_2 = 20\left(\frac{2}{3}\right) + 20\left(\frac{2}{3}\right)^2$

$h_3 = 20\left(\frac{2}{3}\right)^3$ $\qquad \frac{1}{2}S_3 = 20\left(\frac{2}{3}\right) + 20\left(\frac{2}{3}\right)^2 + 20\left(\frac{2}{3}\right)^3$

$h_4 = 20\left(\frac{2}{3}\right)^4$ $\qquad \frac{1}{2}S_4 = 20\left(\frac{2}{3} + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 + \left(\frac{2}{3}\right)^4\right)$

b) $\lim_{N \to \infty} \frac{1}{2}S_N = \sum_{N=0}^{\infty} \underbrace{20}_{a} \underbrace{\left(\frac{2}{3}\right)}_{r}^{N}$

$= \dfrac{20}{1 - \frac{2}{3}}$

$= \dfrac{20}{1/3}$

$= 60$

so the ball travels a total of $2 \times 60 = 120m$.

(1.3) Dusty begins a savings plan in which he deposits \$100 at the beginning of each month into an account that earns 9% interest annually. Let B_n be the balance in the account after the nth payment, where $B_0 = \$0$.

a.) Write the first five terms of the sequence $\{B_n\}$.

b.) Find a recurrence relation that generates the sequence $\{B_n\}$.

c.) How many months are needed to reach a balance of \$5000?

Hint: You can check this by using the time-value of money solver on your Texan friend (Apps -> Finance -> TVM Solver)

(a.) $B_0 = 0$

$B_1 = 100$

$B_2 = 100(1.0075) + 100$

$B_3 = 100(1.0075)^2 + 100(1.0075) + 100$

$B_4 = 100(1.0075)^3 + 100(1.0075)^2 + 100(1.0075) + 100$

(b) $B_0 = 0$

$B_{n+1} = 1.0075 B_n + 100$ for $n = 0, 1, 2, \ldots$

(c.) Texan style: It will take 43 months.

(1.4) Express the sequence $\{8n+5\}_{n=1}^{\infty}$ as an equivalent sequence of the form $\{b_n\}_{n=5}^{\infty}$.

$$\{8(n-4)+5\}_{n=5}^{\infty}$$

OR $$\{8n-27\}_{n=5}^{\infty}$$

(1.5) Express the sequence $\{n^2+5n-3\}_{n=1}^{\infty}$ as an equivalent sequence of the form $\{b_n\}_{n=3}^{\infty}$.

$$\{(n-2)^2+5(n-2)-3\}_{n=3}^{\infty}$$

OR $$\{n^2+n-9\}_{n=3}^{\infty}$$

(1.6) Does the series $\sum_{k=0}^{\infty} \frac{1}{(4k+3)(4k+7)}$ converge or diverge? If the series converges, what is its sum?

$$\sum_{k=0}^{\infty} \frac{1}{(4k+3)(4k+7)}$$

$$= \sum_{k=0}^{\infty}\left(\frac{1/4}{4k+3} - \frac{1/4}{4k+7}\right)$$

$$= \frac{1}{4}\left(\frac{1}{3} - \cancel{\frac{1}{7}} + \cancel{\frac{1}{7}} - \cancel{\frac{1}{11}} + \cancel{\frac{1}{11}} - \cancel{\frac{1}{15}} + \cdots\right)$$

$$= \frac{1}{12}$$

Scratch

$$\frac{1}{(4k+3)(4k+7)} = \frac{A}{4k+3} + \frac{B}{4k+7}$$

$$\Rightarrow 1 = A(4k+7) + B(4k+3)$$

$$\Rightarrow A = \frac{1}{4} \quad \text{and} \quad B = -\frac{1}{4}$$

(1.7) Evaluate the series $\sum_{k=2}^{\infty}\left(\frac{4}{5}\right)^{2k}$ or state that it diverges.

$$= \sum_{k=2}^{\infty}\left(\frac{16}{25}\right)^{k} = \left(\frac{16}{25}\right)^{2} + \left(\frac{16}{25}\right)^{3} + \dots$$

$$a = \left(\frac{16}{25}\right)^{2} \text{ and } r = \frac{16}{25}$$

$$= \frac{\left(\frac{16}{25}\right)^{2}}{1 - \frac{16}{25}} = \frac{\frac{256}{625}}{\frac{9}{25}} = \frac{256}{225}$$

(1.8) Evaluate the series $\sum_{k=1}^{\infty}\left[\frac{1}{7}\left(\frac{3}{4}\right)^{k} + \frac{3}{5}\left(\frac{2}{9}\right)^{k}\right]$ or state that it diverges.

$$= \left(\frac{1}{7}\left(\frac{3}{4}\right) + \frac{3}{5}\left(\frac{2}{9}\right)\right) + \left(\frac{1}{7}\left(\frac{3}{4}\right)^{2} + \frac{3}{5}\left(\frac{2}{9}\right)^{2} + \dots\right)$$

$$= \left(\frac{1}{7}\left(\frac{3}{4}\right) + \frac{1}{7}\left(\frac{3}{4}\right)^{2} + \dots\right) + \left(\frac{3}{5}\left(\frac{2}{9}\right) + \frac{3}{5}\left(\frac{2}{9}\right)^{2} + \dots\right)$$

$$a = \frac{1}{7}\left(\frac{3}{4}\right);\ r = \frac{3}{4} \qquad a = \frac{3}{5}\left(\frac{2}{9}\right);\ r = \frac{2}{9}$$

$$= \frac{\frac{3}{28}}{1 - \frac{3}{4}} + \frac{\frac{6}{45}}{1 - \frac{2}{9}}$$

$$= \frac{3}{28} \cdot \frac{4}{1} + \frac{6}{\cancel{45}_{5}} \cdot \frac{\cancel{9}}{7}$$

$$= \frac{3}{7} + \frac{6}{35}$$

$$= \frac{21}{35}$$

$$= \frac{3}{5}$$

(1.9) Determine how many terms of the series $\sum_{k=0}^{\infty}(0.2)^k$ are needed so that the partial sum is within 10^{-6} of the value of the series (round up to the nearest whole number).

$$R_N < 10^{-6}$$

$$\Rightarrow \sum_{k=N}^{\infty} (0.2)^k < 10^{-6}$$

$$\Rightarrow 0.2^N + 0.2^{N+1} + \dots < 10^{-6}$$

$$a = 0.2^N ; r = 0.2$$

$$\Rightarrow \frac{0.2^N}{1-0.2} < 10^{-6}$$

$$\Rightarrow 0.2^N < 0.8(10^{-6})$$

$$\Rightarrow N \ln(0.2) < \ln(8\times 10^{-7})$$

$$\Rightarrow N > \frac{\ln(8\times 10^{-7})}{\ln(0.2)}$$

$$\Rightarrow N > 8.72$$

Thus 9 terms of the series are needed and $\sum_{k=0}^{8}(0.2)^k$ is w/in 10^{-6} of the infinite sum.

(1.10) Write the repeating decimal $3.\overline{26}$ as a rational number (ratio of two integers).

$$3.\overline{26} = 3 + 0.26 + 0.0026 + 0.000026 + \dots$$

$$= 3 + \frac{26}{100} + \frac{26}{100^2} + \frac{26}{100^3} + \dots \qquad a = \frac{26}{100} ; r = \frac{1}{100}$$

$$= 3 + \frac{\frac{26}{100}}{1-\frac{1}{100}}$$

$$= 3 + \frac{26}{99}$$

$$= \frac{323}{99}$$