Name: Key

Assessment 3

Math& 264: Multivariable Calculus

Instructions: Please carefully complete these questions by hand. Be sure to show all work including sketching relevant graphs and providing exact answers.

Should you choose to work these on scratch paper, please do not put more than one question on a page. Additional sheets of paper are acceptable.

Upload your solutions to Gradescope by 9 am on Monday (4/27). During your presentation time, you will be asked to explain your thought process and reasoning on one randomly assigned question.

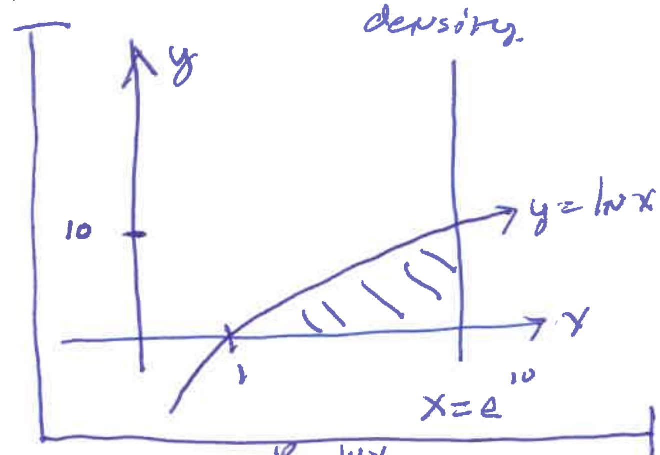
Please make sure to sign up for your presentation slot. If you are unavailable for any of the times available, please send me a note in Slack and we will find a time that works for you.

https://docs.google.com/spreadsheets/d/1IGUR3J5qnXkbLUhWWeoMjbxYb1peN9lt5gZYj7wAtCo/edit?usp=sharing

(1.1) Find the centroid of the thin plate bounded by $y=\ln(x)$, the x-axis, and $x=e^{10}$. with uniform density.

$$m = \int_1^{e^{10}}\int_0^{\ln x} 1\,dy\,dx = \int_1^{e^{10}} \ln x\,dx = \Big[x\ln x - x\Big]_1^{e^{10}}$$

$$= 10e^{10} - e^{10} + 1 = 9e^{10}+1$$



$$M_y = \int_1^{e^{10}}\int_0^{\ln x} x\,dy\,dx = \int_1^{e^{10}} x\ln x\,dx = \Big[\tfrac{1}{4}x^2(2\ln x - 1)\Big]_1^{e^{10}}$$

$$= \tfrac{1}{4}e^{20}(20-1) - \tfrac{1}{4}(-1) = \tfrac{1}{4}(19e^{20}+1)$$

$$M_x = \int_1^{e^{10}}\int_0^{\ln x} y\,dy\,dx = \frac{1}{2}\int_1^{e^{10}} (\ln x)^2\,dx$$

$$= \frac{1}{2}\Big[x(\ln x)^2 - 2x\ln x + 2x\Big]$$

$$= \frac{1}{2}\Big[(e^{10}\cdot 100 - 2e^{10}\cdot 10 + 2e^{10}) - (0 - 0 + 2)\Big]$$

$$= 41e^{10} - 1$$

$$\bar{x} = \frac{19e^{20}+1}{4(9e^{10}+1)} \approx 11625 \quad \text{and} \quad \bar{y} = \frac{41e^{10}-1}{9e^{10}+1} \approx 4.56$$

(1.2) Solve the relations $u = 24xy$ and $v = 8x$ for x and y, and compute the Jacobian.

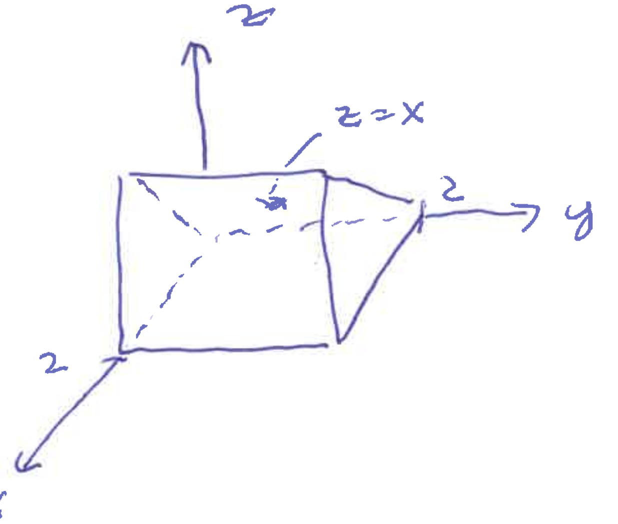
$$v = 8x \Rightarrow x = \frac{v}{8}$$

$$u = 24xy = 24 \cdot \frac{v}{8} y \Rightarrow y = \frac{u}{3v}$$

$$\Rightarrow \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 0 & \frac{1}{8} \\ \frac{1}{3v} & -\frac{u}{3v^2} \end{vmatrix}$$

$$= -\frac{1}{24v}$$

(1.3) Find the center of mass of the prism formed by $z = x$, $x = 2$, $y = 2$, and the coordinate planes with variable density function $\rho(x,y,z) = 3 + y$.



$$m = \int_0^2 \int_0^2 \underbrace{\int_0^x (3+y)\,dz}_{\left[3z + yz\right]_{z=0}^{z=x}}\,dx\,dy$$

$$= \int_0^2 \underbrace{\int_0^2 (3x + yx)\,dx}_{\left[\frac{3}{2}x^2 + \frac{1}{2}yx^2\right]_0^2}\,dy$$

$$= \int_0^2 (6 + 2y)\,dy$$

$$= \left[6y + y^2\right]_0^2$$

$$= 16$$

$$M_{xz} = \int_0^2\int_0^2\int_0^x y(3+y)\,dz\,dx\,dy$$

$$= \int_0^2\int_0^2 x\,(3y + y^2)\,dx\,dy$$

$$= \frac{1}{2}\int_0^2 4(3y + y^2)\,dy$$

$$= 2\left[\frac{3}{2}y^2 + \frac{1}{3}y^3\right]_0^2$$

$$= \frac{52}{3}$$

$$M_{xy} = \int_0^2\int_0^2\int_0^x z(3+y)\,dz\,dx\,dy$$

$$= \frac{1}{2}\int_0^2\int_0^2 x^2(3+y)\,dx\,dy$$

$$= \frac{1}{2}\cdot\frac{8}{3}\int_0^2 (3+y)\,dy$$

$$= \frac{4}{3}\left[3y + \frac{y^2}{2}\right]_0^2$$

$$= \frac{32}{3}$$

$$M_{yz} = \int_0^2 \int_0^2 \int_0^x x(3+y)\,dz\,dx\,dy$$

$$= \int_0^2 \int_0^2 x^2(3+y)\,dx\,dy$$

$$= \frac{8}{3} \int_0^2 (3+y)\,dy$$

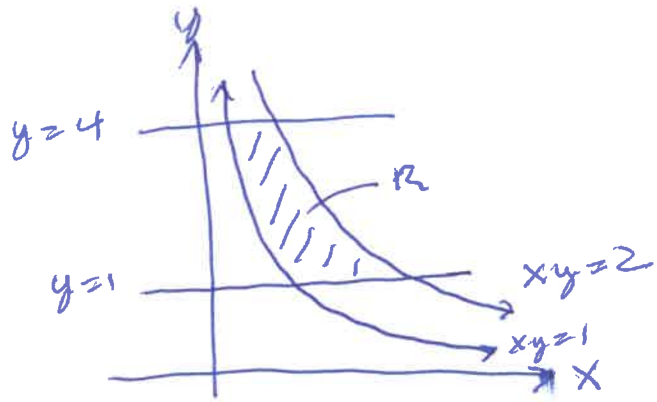
$$= \frac{8}{3} \left[3y + \frac{1}{2} y^2 \right]_0^2$$

$$= \frac{64}{3}$$

$$(\bar{x}, \bar{y}, \bar{z}) = \left(\frac{\frac{64}{3}}{16}, \frac{\frac{52}{3}}{16}, \frac{\frac{32}{3}}{16} \right)$$

$$= \left(\frac{4}{3}, \frac{13}{12}, \frac{2}{3} \right)$$

(1.4) Evaluate the integral $I = \iint_R xy\, DA$ where R is the region bounded by the hyperbolas $xy = 1$, $xy = 2$, and the lines $y = 1$ and $y = 4$.



Let $u = xy$ and $v = y$

$$\Rightarrow \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} y & x \\ 0 & 1 \end{vmatrix} = y$$

$$\Rightarrow \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{y} = \frac{1}{v}$$

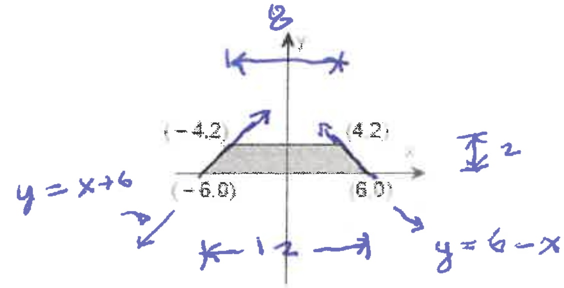
$$\Rightarrow I = \int_1^4 \int_1^2 \underbrace{u \cdot \left|\frac{1}{v}\right|}_{\frac{u}{v} \text{ since } 1 \leq v \leq 4} \, du\, dv$$

$$= \int_1^4 \left[\frac{1}{2} u^2 \cdot \frac{1}{v}\right]_1^2 dv$$

$$= \frac{3}{2}\left[\ln|v|\right]_1^4$$

$$= \frac{3}{2}\ln(4) \text{ or } 3\ln 2$$

(1.5) Find the mass and centroid of the thin constant-density plate shown assuming density of 1.



$$M = \frac{1}{2}\cdot 2(8+12) = 20$$

$$M_x = \underbrace{\int_0^2 \int_{y-6}^{6-y} 1\, y\, dx\, dy}$$

$$\Big[xy\Big]_{x=y-6}^{x=6-y} \qquad (6-y)y - (y-6)y$$

$$= \int_0^2 (12y - 2y^2)\,dy$$

by symmetry, $\bar{x} = 0$

$$= \Big[6y^2 - \frac{2}{3}y^3\Big]_0^2$$

$$= 24 - \frac{16}{3}$$

$$= \frac{56}{3}$$

$$\Rightarrow (\bar{x}, \bar{y}) = \left(0, \frac{56}{60}\right) = \left(0, \frac{14}{15}\right)$$

(1.6) Evaluate $I = \iiint_D xy\,dV$ where D is bounded by the planes $y-x=0$, $y-x=2$, $z-y=0$, $z-y=1$, $z=0$, and $z=8$.

Let $u = y-x$, $v = z-y$, and $w = z$

$$\Rightarrow \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 1$$

$$\Rightarrow \frac{\partial(x,y,z)}{\partial(u,v,w)} = 1$$

Integrand: xy

$$v = w - y \Rightarrow y = w - v$$

$$u = (w-v) - x \Rightarrow x = w - v - u$$

$$\Rightarrow xy = (w-v)(w-v-u)$$

$$= w^2 - 2vw + v^2 - uw + uv$$

$$I = \int_0^8 \int_0^1 \int_0^2 \underbrace{(w^2 - 2vw + v^2 - uw + uv)\,du}\,dv\,dw \quad \text{scaling factor} = 1$$

$$= \int_0^8 \int_0^1 2w^2 - 4vw + 2v^2 - 2w + 2v \; dv\,dw$$

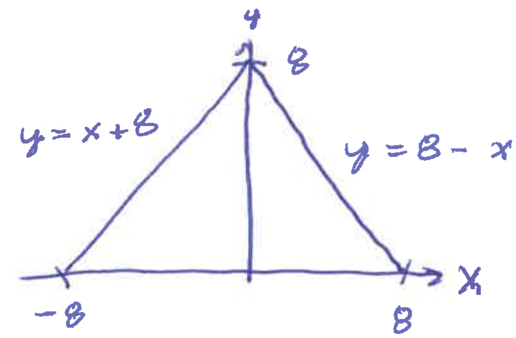
$$= \int_0^8 2w^2 - 2w + \frac{2}{3} - 2w + 1 \; dw$$

$$= \int_0^8 \left(2w^2 - 4w + \frac{5}{3}\right) dw$$

$$= \frac{2}{3}8^3 - 2\cdot 8^2 + \frac{5}{3}\cdot 8$$

$$= \frac{680}{3}$$

(1.7) Find the mass and centroid of the plate bounded by $y = 8 - |x|$ and the x-axis. Assume density 1.



$\bar{X} = 0$

$M = 64$

$$M_x = \int_0^8 \int_{y-8}^{8-y} y \, dx \, dy$$

$$= \int_0^8 \Big[xy \Big]_{y-8}^{8-y} dy$$

$$= \int_0^8 16y - 2y^2 \, dy$$

$$= \Big[8y^2 - \frac{2}{3} y^3 \Big]_0^8$$

$$= \Big[\frac{1}{3} y^3 \Big]_0^8$$

$$= \frac{512}{3}$$

$$(\bar{X}, \bar{y}) = \left(0, \frac{\frac{512}{3}}{64}\right) = \left(0, \frac{8}{3}\right)$$

(1.8) Let R be the region bounded by the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, where $a>0$ and $b>0$ are real numbers.

Let T be the transformation $x = au$ and $y = bv$. Evaluate $I = \iint_R |xy|\, dA$.

$$\Rightarrow \quad \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix} = ab$$

$$\Rightarrow \quad I = 4 \iint ab\, uv \cdot ab \; dA'$$

$$= 4a^2b^2 \int_0^{\frac{\pi}{2}} \int_0^1 r^2 \sin\theta \cos\theta \cdot r \; dr\, d\theta$$

$$= 4a^2b^2 \underbrace{\int_0^{\pi/2} \sin\theta\cos\theta\, d\theta}_{} \; \underbrace{\int_0^1 r^3\, dr}_{}$$

$$= 4a^2b^2 \left[\frac{1}{2}\sin^2\theta\right]_0^{\frac{\pi}{2}} \qquad \frac{1}{4}$$

$$= \frac{1}{2}a^2b^2$$

(1.9) Find the center of mass of the upper half $(y \geq 0)$ of the plate bounded by the ellipse $x^2 + 9y^2 = 9$ with density $\rho(x,y) = 1 + y$.

Let $u = x$ and $v = 3y$

$$\Rightarrow \quad \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1 & 0 \\ 0 & 3 \end{vmatrix} = 3$$

$$\Rightarrow \quad \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{3}$$

$u = r\cos\theta$

$v = r\sin\theta$

$$m = \iint \left(1 + \frac{v}{3}\right) \cdot \frac{1}{3} \, dA'$$

$$= \int_0^{\pi}\int_0^{3} \left(1 + \frac{r\sin\theta}{3}\right)\frac{1}{3} \, r \, dr \, d\theta$$

$$= \frac{1}{9}\int_0^{\pi}\int_0^{3} (3r + r^2\sin\theta) \, dr \, d\theta$$

$$= \frac{1}{9}\int_0^{\pi} \left[\frac{3}{2}r^2 + \frac{1}{3}r^3\sin\theta\right]_0^3 d\theta$$

$$= \frac{1}{9}\int_0^{\pi} \left(\frac{27}{2} + 9\sin\theta\right) d\theta$$

$$= \frac{1}{9}\left(\frac{27}{2}\pi + 9(2)\right)$$

$$= \frac{3}{2}\pi + 2$$

$$M_x = \frac{1}{9}\int_0^{\pi}\int_0^{3} (3r + r^2\sin\theta)\cdot \frac{\overset{y}{\overbrace{r\sin\theta}}}{3}\, dr\, d\theta$$

$$= \frac{1}{27}\int_0^{\pi}\underbrace{\int_0^{3}(3r^2\sin\theta + r^3\sin^2\theta)\, dr}_{\left[r^3\sin\theta + \frac{r^4}{4}\sin^2\theta\right]_0^3}\, d\theta$$

$$= \frac{1}{27}\int_0^{\pi}\left(27\sin\theta + \frac{81}{4}\sin^2\theta\right)d\theta$$

$$= 2 + \frac{3}{4}\int_0^{\pi}\frac{1-\cos 2\theta}{2}\, d\theta$$

$$= 2 + \frac{3}{8}\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\pi}$$

$$= 2 + \frac{3}{8}\pi$$

$$(\bar{x}, \bar{y}) = \left(0, \frac{2 + \frac{3}{8}\pi}{\frac{3}{2}\pi + 2}\right) = \left(0, \frac{16 + 3\pi}{4(3\pi + 4)}\right)$$

(1.10) Let D be the solid bounded by the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, where $a>0$, $b>0$, and $c>0$ are real numbers. Let T be the transformation $x=au$, $y=bv$, and $z=cw$. Evaluate $I=\iint\limits_D |xyz|\,dV$.

$$\Rightarrow \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc$$

$$\Rightarrow I = 8\iiint (abc)^2 uvw \, dV'$$

include scaling factor

$$= 8(abc)^2 \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^1 \rho\cos\theta\sin\phi \; \rho\sin\theta\sin\phi \; \rho\cos\phi \; \rho^2\sin\phi \, d\rho\, d\phi\, d\theta$$

$$= 8(abc)^2 \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^1 \rho^5 \sin\theta\cos\theta \sin^3\phi\cos\phi \, d\phi$$

$$= 8(abc)^2 \left[\frac{\rho^6}{6}\right]_0^1 \left[\frac{1}{2}\sin^2\theta\right]_0^{\pi/2} \left[\frac{1}{4}\sin^4\phi\right]_0^{\pi/2}$$

$$= 8(abc)^2 \cdot \frac{1}{6} \cdot \frac{1}{2} \cdot \frac{1}{4}$$

$$= \frac{(abc)^2}{6}$$