Name: Key.

Assessment 2

Math& 264: Multivariable Calculus

Instructions: Please carefully complete these questions by hand. Be sure to show all work including sketching relevant graphs and providing exact answers. Upload your solutions to Gradescope by 8 am on Monday (4/20). During your presentation time, you will be asked to explain your thought process and reasoning on one randomly assigned question.

Please make sure to sign up for your presentation slot. If you are unavailable for any of the times available, please send me a note in Slack and we will find a time that works for you.

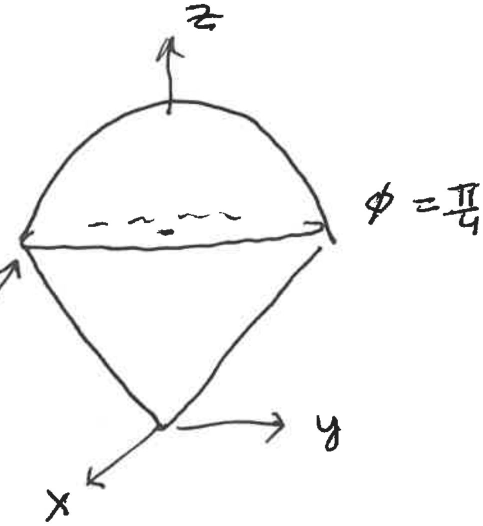
https://docs.google.com/spreadsheets/d/1IGUR3J5qnXkbLUhWWeoMjbxYb1peN9lt5gZYj7wAtCo/edit?usp=sharing

(1.1) Use a triple integral to find the volume of the solid bounded below by the cone $z=\sqrt{x^2+y^2}$ and bounded above by the sphere $x^2+y^2+z^2=72$.

Find the intersection,

$$x^2+y^2+\left(\sqrt{x^2+y^2}\right)^2=72$$

$$\Rightarrow x^2+y^2=36 \quad w/\ z=6$$



Integrate w/spherical coordinates.

$$V=\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{6\sqrt{2}} 1\,\rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$

$$=2\pi\int_0^{\frac{\pi}{4}}\sin\phi\, d\phi\int_0^{6\sqrt{2}}\rho^2\, d\rho$$

$$=2\pi\underbrace{\Big[-\cos\phi\Big]_0^{\frac{\pi}{4}}}_{-\frac{\sqrt{2}}{2}+1}\cdot\underbrace{\Big[\tfrac{1}{3}\rho^3\Big]_0^{6\sqrt{2}}}_{\frac{6^3\cdot 2\sqrt{2}}{3}}$$

$$=2\pi\left(-\frac{\sqrt{2}}{2}+1\right)\checkmark \quad \frac{6^3\cdot 2\sqrt{2}}{3}\checkmark$$

$$=\pi(2-\sqrt{2})\cdot 144\sqrt{2}$$

$$=288\pi(\sqrt{2}-1)$$

(1.2) Evaluate the integral $I = \int_0^6 \int_0^{\sqrt{2}} \int_x^{\sqrt{4-x^2}} e^{-x^2-y^2}\, dy\, dx\, dz$

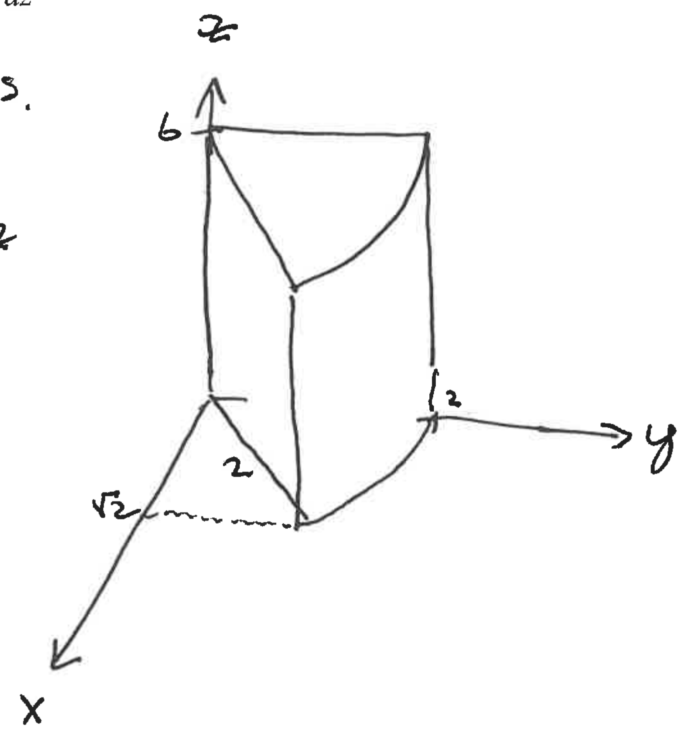
use cylindrical coordinates.

$$I = \int_0^6 \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^2 e^{-r^2}\, r\, dr\, d\theta\, dz$$

$$= 6 \cdot \frac{\pi}{4} \left[-\frac{1}{2} e^{-r^2} \right]_0^2$$

$$= \frac{3}{2}\pi \left(\frac{1}{2} - \frac{1}{2} e^{-4} \right)$$

$$= \frac{3}{4}\pi \left(1 - \frac{1}{e^4} \right)$$



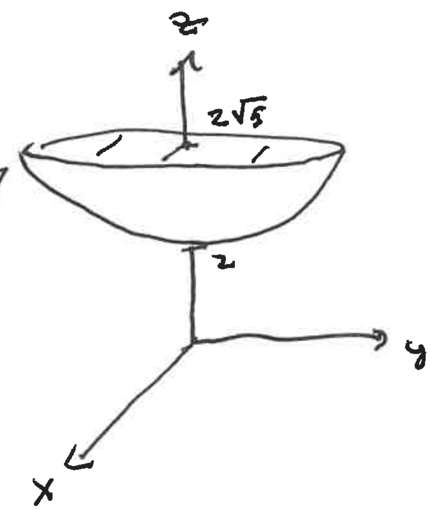
(1.3) Find the volume of the region bounded by the plane $z = 2\sqrt{5}$ and $z = \sqrt{4 + x^2 + y^2}$.

Find the intersection.

$$2\sqrt{5} = \sqrt{4 + x^2 + y^2}$$

$$\Rightarrow 20 = 4 + x^2 + y^2$$

$$\Rightarrow 16 = x^2 + y^2$$



use cylindrical coordinates.

$$V = \int_0^{2\pi}\int_0^4\int_{\sqrt{4+r^2}}^{2\sqrt{5}} 1 \, r \, dz \, dr \, d\theta$$

$$= 2\pi \int_0^4 \Big[r z \Big]_{\sqrt{4+r^2}}^{2\sqrt{5}} dr$$

$$= 2\pi \int_0^4 (2\sqrt{5}\, r - r\sqrt{4+r^2}) \, dr$$

$$= 2\pi \left[\sqrt{5}\, r^2 - \frac{1}{3}(4+r^2)^{\frac{3}{2}} \right]_0^4$$

$$= 2\pi \left(16\sqrt{5} - \frac{1}{3}\left(20^{3/2} - 8\right)\right)$$

$$= \frac{16}{3}\pi (1 + \sqrt{5})$$

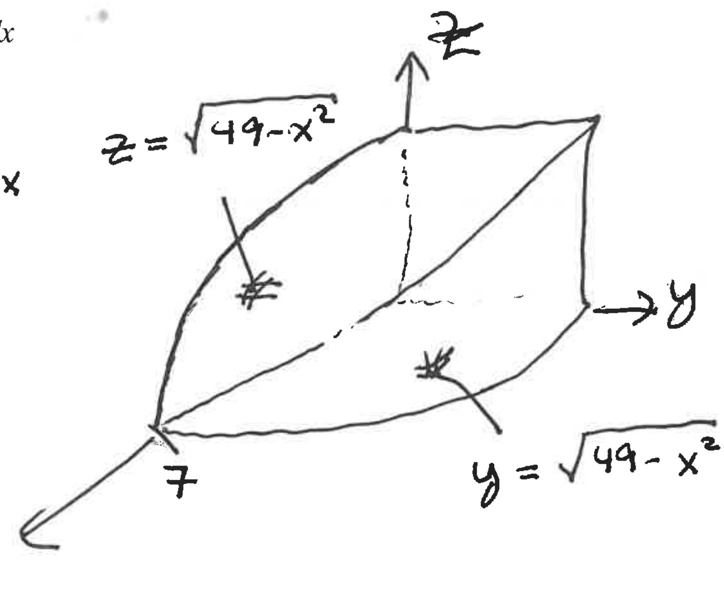
(1.4) Evaluate the integral $I = \int_0^7 \int_0^{\sqrt{49-x^2}} \int_0^{\sqrt{49-x^2}} 1\, dz\, dy\, dx$

$$I = \int_0^7 \int_0^{\sqrt{49-x^2}} \sqrt{49-x^2}\, dy\, dx$$

$$= \int_0^7 49 - x^2\, dx$$

$$= \left[49x - \frac{1}{3}x^3 \right]_0^7$$

$$= \frac{686}{3}$$



$I=$

(1.5) Evaluate the integral $\int_0^{2\pi}\int_0^{\frac{\pi}{6}}\int_0^{2\sec\phi} \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta$

$$I = 2\pi \int_0^{\frac{\pi}{6}} \left[\frac{8}{3} \rho^3 \sin\phi \right]_0^{2\sec\phi} d\phi$$

$$= \frac{2\pi}{3} \int_0^{\frac{\pi}{6}} 8\, \underbrace{\sec^3\phi \, \sin\phi}_{\frac{\sin\phi}{\cos^3\phi}} \, d\phi$$

Let $u = \cos\phi$ and $du = -\sin\phi \, d\emptyset$

w/ $u(0) = 1$ and $u(\frac{\pi}{6}) = \frac{\sqrt{3}}{2}$

$$= \frac{2\pi}{3} \int_1^{\frac{\sqrt{3}}{2}} -8u^{-3} \, du$$

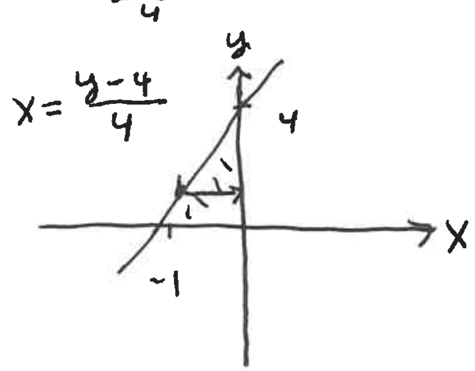
$$= \frac{2\pi}{3} \left[\frac{8}{2} u^{-2} \right]_1^{\frac{\sqrt{3}}{2}}$$

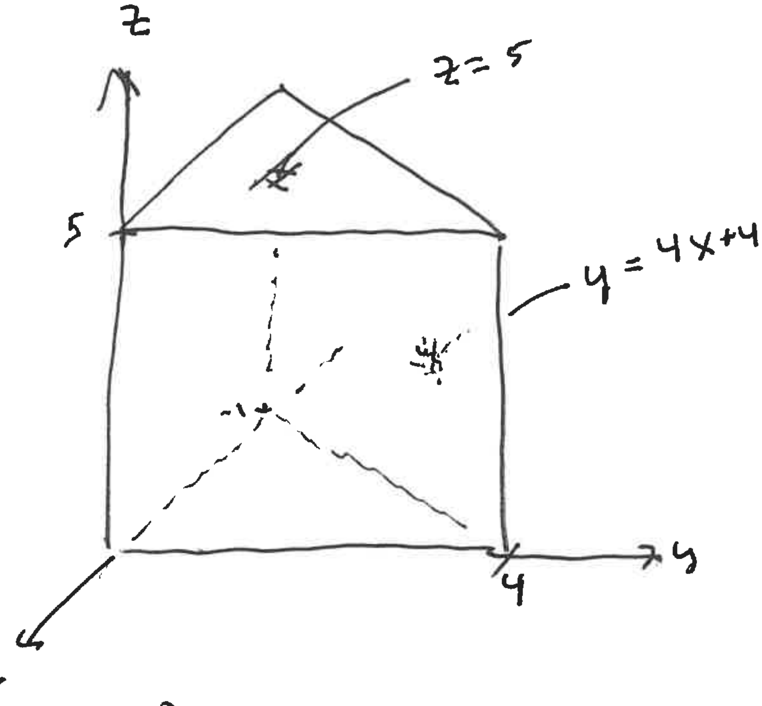
$$= \frac{8\pi}{3} \left(\frac{4}{3} - 1 \right)$$

$$= \frac{8\pi}{9}$$

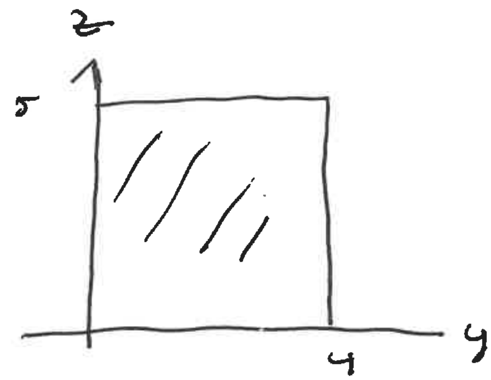
(1.6) Rewrite the integral $I = \int_0^5 \int_{-1}^0 \int_0^{4x+4} 1\, dy\, dx\, dz$ in the orders $dz\, dx\, dy$ and $dx\, dz\, dy$

$$I = \int_0^4 \int_{\frac{y-4}{4}}^0 \int_0^5 1\, dz\, dx\, dy$$





$$I = \int_0^4 \int_0^5 \int_{x=\frac{y-4}{4}}^{x=0} 1\, dx\, dz\, dy$$



$I =$

(1.7) Evaluate the integral $\int_0^{\pi}\int_0^{\frac{\pi}{4}}\int_{2\sec\phi}^{4} \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta$

$$I = \pi \int_0^{\frac{\pi}{4}} \left[\frac{1}{3} \rho^3 \sin\phi \right]_{2\sec\phi}^{4} d\phi$$

$$= \frac{\pi}{3} \int_0^{\frac{\pi}{4}} (64 - 8\sec^3\phi) \sin\phi \, d\phi$$

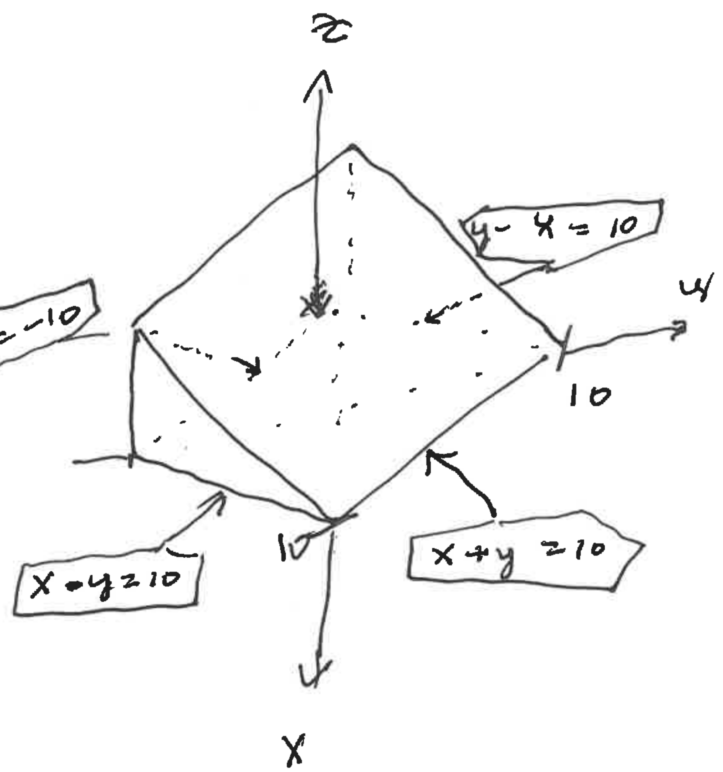
$$= \frac{8\pi}{3} \int_0^{\pi/4} 8\sin\phi - \frac{\sin\phi}{\cos^3\phi} \, d\phi$$

$$= \frac{8\pi}{3} \left[-8\cos\phi - \frac{1}{2} \frac{1}{\cos^2\phi} \right]_0^{\pi/4}$$

$$= \frac{8\pi}{3} \left(-8\frac{\sqrt{2}}{2} - \frac{1}{2} \cdot \frac{4}{2} + 8 + \frac{1}{2} \right)$$

$$= \frac{\pi}{3}(60 - 32\sqrt{2})$$

(1.8) Use a triple integral to compute the volume of the wedge of the square column $|x|+|y|=10$ created by the planes $z=0$ and $x+y+z=10$.



$$V = \int_0^{10}\int_{x-10}^{10-x}\int_0^{10-x-y} 1\,dz\,dy\,dx$$

$$+$$

$$\int_{-10}^{0}\int_{-x-10}^{x+10}\int_0^{10-x-y} 1\,dz\,dy\,dx$$

$$= \int_0^{10}\int_{x-10}^{10-x} (10-x-y)\,dy\,dx$$

$$+$$

$$\int_{-10}^{0}\int_{-x-10}^{x+10} (10-x-y)\,dy\,dx$$

$$= \int_0^{10} \left[10y - xy - \tfrac{1}{2}y^2\right]_{x-10}^{10-x} dx + \int_{-10}^{0} \left[10y - xy - \tfrac{1}{2}y^2\right]_{-x-10}^{x+10} dx$$

$$= \int_0^{10} 200 - 40x + 2x^2 - \tfrac{1}{2}\underbrace{\left[(10-x)^2 - (x-10)^2\right]}_{0} dx$$

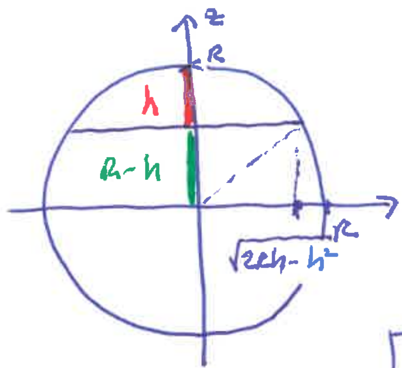
$$+ \int_{-10}^{0} 10(2x+20) - x(2x+20) - \tfrac{1}{2}\underbrace{\left[(x+10)^2 - (-x-10)^2\right]}_{0} dx$$

$$= \int_0^{10} 200 - 40x + 2x^2\,dx + \int_{-10}^{0} 200 - 2x^2\,dx$$

$$= \left[200x - 20x^2 + \tfrac{2}{3}x^3\right]_0^{10} + \left[200x - \tfrac{2}{3}x^3\right]_{-10}^{0}$$

$$= 2000 - 2000 + \frac{2000}{3} + 2000 - \frac{2000}{3} = \boxed{2000}$$

(1.9) Use multiple integrals to find the volume of the cap of a sphere of radius R with thickness h. Assume that R and h are positive constants.

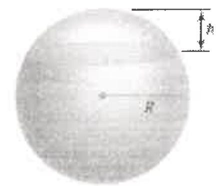


$$x^2 + y^2 + z^2 = R^2$$

$$\uparrow \quad (R-h)$$

$$\Rightarrow x^2 + y^2 = -(R-h)^2 + R^2$$

$$\Rightarrow r^2 = 2Rh - h^2$$



$$\boxed{\text{using cylindrical coordinates}}$$

$$V = \int_0^{2\pi} \int_0^{\sqrt{2Rh-h^2}} \int_{R-h}^{\sqrt{R^2-r^2}} 1 \, r \, dz \, dr \, d\theta$$

$$= 2\pi \int_0^{\sqrt{2Rh-h^2}} r\sqrt{R^2-r^2} - r(R-h) \, dr$$

$$= 2\pi \left[-\frac{1}{3}(R^2-r^2)^{3/2} - \frac{1}{2}r^2(R-h) \right]_0^{\sqrt{2Rh-h^2}}$$

$$= 2\pi \left[\left(-\frac{1}{3}(R^2-2Rh+h^2)^{3/2} - \frac{1}{2}(2Rh-h^2)(R-h) \right) - \left(-\frac{1}{3}R^3 - 0 \right) \right]$$

$$= 2\pi \left(\frac{1}{3}R^3 - \frac{1}{3}(R-h)^3 - \frac{1}{2}(R-h)(2Rh-h^2) \right)$$

$$= 2\pi \left(\cancel{\frac{1}{3}R^3} - \cancel{\frac{1}{3}R^3} + \cancel{R^2h} - \cancel{Rh^2} + \frac{1}{3}h^3 - \cancel{R^2h} + \frac{1}{2}Rh^2 + \cancel{Rh^2} - \frac{1}{2}h^3 \right)$$

$$= 2\pi \left(\frac{1}{2}Rh^2 - \frac{1}{6}h^3 \right)$$

$$= \frac{1}{3}\pi h^2 (3R - h)$$

$I =$

(1.10) Rewrite the integral $\int_0^4 \int_0^{\sqrt{16-x^2}} \int_0^{\sqrt{16-x^2}} 1\, dy\, dz\, dx$ in the order $dz\, dy\, dx$

$$I = \int_0^4 \int_0^{\sqrt{16-x^2}} \int_0^{\sqrt{16-x^2}} 1\, dz\, dy\, dx$$