Name: Key

Assessment 1

Math& 264: Multivariable Calculus

Instructions: Please carefully complete these questions by hand making sure to show all work. Upload your solutions to Gradescope by 8 am on Monday (4/13). During your presentation time, you will be asked to explain your thought process and reasoning on one randomly assigned question.

(1.1) Convert the equation $r = \dfrac{1}{2\cos\theta + 3\sin\theta}$ to Cartesian coordinates. Describe the resulting curve.

$x = r\cos\theta$

$y = r\sin\theta$

$$\Rightarrow r = \frac{1}{\frac{2x}{r} + \frac{3y}{r}}$$

$$\Rightarrow r = \frac{r}{2x + 3y}$$

$$\Rightarrow 1 = \frac{1}{2x + 3y}$$

$$\Rightarrow 2x + 3y = 1$$

$$\Rightarrow y = \frac{1 - 2x}{3}$$ This is a line.

(1.2) Find the slope of the line tangent to $r = 8\sin\theta$ at the point $\left(4, \dfrac{5\pi}{6}\right)$

$r = 8\sin\theta$

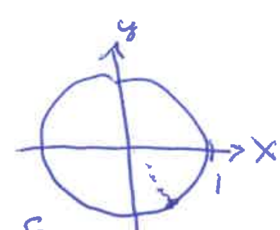
$r' = 8\cos\theta$

$$\frac{dy}{dx} = \frac{r'\sin\theta + r\cos\theta}{r'\cos\theta - r\sin\theta}$$

$$= \frac{8\cos\theta\sin\theta + 8\sin\theta\cos\theta}{8\cos^2\theta - 8\sin^2\theta}$$

$$= \frac{8\sin 2\theta}{8\cos 2\theta}$$

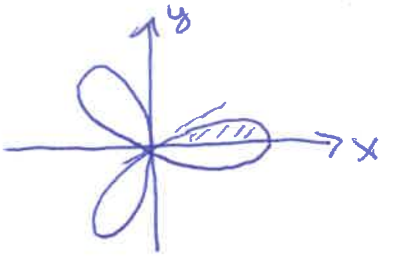
$$= \frac{\sin\left(\frac{5\pi}{3}\right)}{\cos\left(\frac{5\pi}{3}\right)}$$



$$\frac{dy}{dx} = \frac{-\frac{\sqrt{3}}{2}}{\frac{1}{2}} = -\sqrt{3}$$

The slope is $-\sqrt{3}$

(1.3) Find the area of the region inside one leaf of $r = \cos(3\theta)$



$$\text{Area} = 2 \cdot \int_0^{\pi/6} \frac{1}{2}\cos^2(3\theta)\,d\theta$$

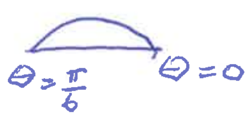
$$= \int_0^{\pi/6} \frac{1+\cos 6\theta}{2}\,d\theta$$

$$= \frac{1}{2}\left[\theta + \frac{1}{6}\sin 6\theta\right]_{\theta=0}^{\theta=\pi/6}$$

$$= \frac{1}{2}\cdot\frac{\pi}{6}$$

$$= \frac{\pi}{12}$$

Area = 2 *

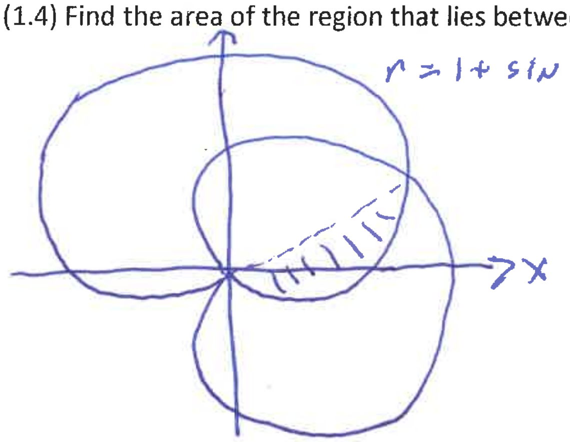


Solve $\cos 3\theta = 0$

$\Rightarrow 3\theta = \frac{\pi}{2} + k\pi$

$\Rightarrow \theta = \frac{\pi}{6} + k\frac{\pi}{3}$

(1.4) Find the area of the region that lies between $r = 1 + \sin\theta$ and $r = 1 + \cos\theta$



<u>Solve</u> $1 + \sin\theta = 1 + \cos\theta$

$\Rightarrow \sin\theta = \cos\theta$

$\Rightarrow \theta = \frac{\pi}{4}, \frac{5\pi}{4}$

$$\text{Area} = 2 \cdot \int_{\pi/4}^{5\pi/4} \frac{1}{2}(1+\cos\theta)^2\,d\theta$$

$$= \int_{\pi/4}^{5\pi/4} (1 + 2\cos\theta + \underbrace{\cos^2\theta}_{\frac{1+\cos 2\theta}{2}})\,d\theta$$

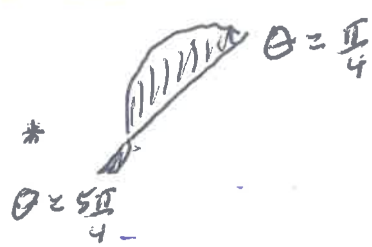
$$= \int_{\pi/4}^{5\pi/4} \left(\frac{3}{2} + 2\cos\theta + \frac{1}{2}\cos 2\theta\right)d\theta$$

$$= \left[\frac{3}{2}\theta + 2\sin\theta + \frac{1}{4}\sin 2\theta\right]_{\pi/4}^{5\pi/4}$$

$$= \frac{3}{2}\cdot\frac{5\pi}{4} - \sqrt{2} + \frac{1}{4} - \frac{3\pi}{8} - \sqrt{2} - \frac{1}{4}$$

$$= \frac{3}{2}\pi - 2\sqrt{2}$$

Area = 2 *



(1.5) Find the length of the complete cardioid $r = 4 + 4\sin\theta$

$$r' = 4\cos\theta$$

$$L = 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{16\cos^2\theta + 16(1+\sin\theta)^2}\, d\theta$$

$$= 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 4\sqrt{\cos^2\theta + 1 + \sin^2\theta + 2\sin\theta}\; d\theta$$

$$= 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 4\sqrt{2}\sqrt{1+\sin\theta}\; d\theta$$

$$= 8\sqrt{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{\frac{(1+\sin\theta)(1-\sin\theta)}{1-\sin\theta}}\, d\theta$$

$$= 8\sqrt{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\theta\sqrt{\frac{1}{1-\sin\theta}}\, d\theta$$

let $u = 1 - \sin\theta$
$du = -\cos\theta\, d\theta$

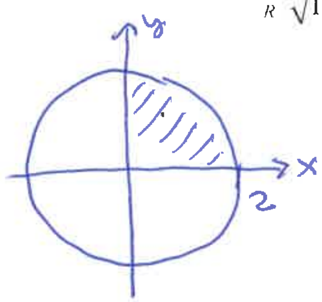
$$\Rightarrow\; = -8\sqrt{2}\int_{2}^{0} u^{-1/2}\, du$$

$$= -8\sqrt{2}\left[2u^{1/2}\right]_2^0$$

$$= -8\sqrt{2}\,(2\sqrt{2})$$

$$= 32.$$

(1.6) Evaluate $I = \iint_R \frac{dA}{\sqrt{16 - x^2 - y^2}}$ where $R = \{(x,y) : x^2 + y^2 \le 4, x \ge 0, y \ge 0\}$



$$I = \int_0^{\frac{\pi}{2}}\int_0^2 \frac{1}{\sqrt{16 - r^2}}\, r\, dr\, d\theta$$

let $u = 16 - r^2$

$du = -2r\, dr$

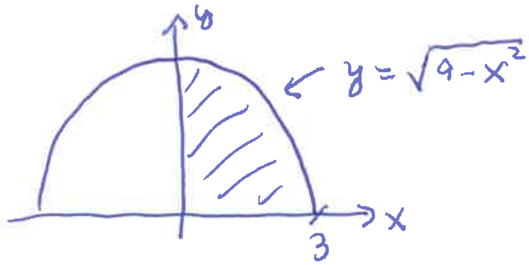
$$\Rightarrow I = \frac{\pi}{2}\int_{16}^{12} -\frac{1}{2}u^{-1/2}\, du$$

$$\Rightarrow I = -\frac{\pi}{4}\left[2u^{1/2}\right]_{16}^{12}$$

$$= \frac{\pi}{2}(4 - \sqrt{12})$$

$$= 2\pi - \sqrt{3}\pi$$

(1.7) Evaluate $I = \int_0^3 \int_0^{\sqrt{9-x^2}} \sqrt{x^2+y^2}\, dy\, dx$



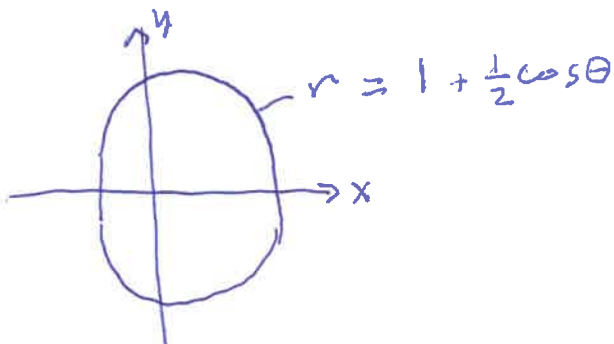
$$\Rightarrow \quad I = \int_0^{\frac{\pi}{2}} \int_0^3 \sqrt{r^2}\; r\, dr\, d\theta$$

$$= \frac{\pi}{2} \int_0^3 r^2\, dr$$

$$= \frac{\pi}{2} \cdot \left[\frac{1}{3} r^3 \right]_0^3$$

$$= \frac{9}{2}\pi$$

(1.8) Sketch and set up an iterated integral representing the area of the region inside the limaçon $r = 1 + \frac{1}{2}\cos\theta$



$$\text{Area} = \int_0^{2\pi} \int_0^{1+\frac{1}{2}\cos\theta} 1 \; r\, dr\, d\theta$$

(1.9) Use a double integral to find the area of the region bounded by the cardioid $r = 2(1-\sin\theta)$

$$\text{Area} = \int_0^{2\pi}\int_0^{2(1-\sin\theta)} 1\, r\, dr\, d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} (2(1-\sin\theta))^2\, d\theta$$

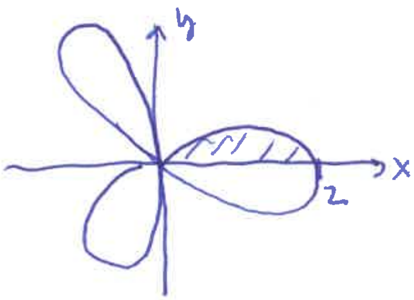
$$= 2\int_0^{2\pi} (1 - 2\sin\theta + \underbrace{\sin^2\theta}_{\frac{1-\cos 2\theta}{2}})\, d\theta$$

$$= 2\int_0^{2\pi} (\tfrac{3}{2} - 2\sin\theta - \cos 2\theta)\, d\theta$$

$$= 2\left[\tfrac{3}{2}\theta + 2\cos\theta - \tfrac{1}{2}\sin 2\theta\right]_0^{2\pi}$$

$$= 6\pi$$

(1.10) Use a double integral to find the area of the region bounded by all leaves of $r = 2\cos(3\theta)$



Area = 6 *

$\theta = \frac{\pi}{6}$ $\theta = 0$

$$= 6\int_0^{\frac{\pi}{6}}\int_0^{2\cos(3\theta)} 1\, r\, dr\, d\theta$$

$$= 6\cdot\frac{1}{2}\int_0^{\pi/6} (2\cos 3\theta)^2\, d\theta$$

$$= 3\cdot 4\int_0^{\pi/6} \cos^2(3\theta)\, d\theta$$

$$= 6\int_0^{\pi/6} 1 + \cos(6\theta)\, d\theta$$

$$= 6\left[\theta + \tfrac{1}{6}\sin 6\theta\right]_0^{\pi/6}$$

$$= \pi$$