HW: 32 min + 5 Q's.

The Divergence Theorem (Gauss's Theorem)

Vector fields can represent electric or magnetic fields, air velocities in hurricanes, or blood flow in an artery. These and other vector phenomena suggest movement of a "substance." A frequent question concerns the amount of a substance that flows across a surface - for example, the amount of water that passes across the membrane of a cell per unit time. Such flux calculations may be done using flux integrals as done previously. The Divergence Theorem offers an alternative method. In effect, it says that instead of integrating the flow in and out of a region across its boundary, you may also add up all the sources (or sinks) of the flow throughout the region.

Throughout this chapter, we have been drawing parallels between theorems, ~~and~~ the divergence theorem is no different.

- The circulation form of Green's Theorem → Stokes' Theorem

$$\text{2D: circulation} = \oint_C \vec{F}\cdot\vec{T}\,ds = \iint_R \underbrace{\left(\frac{\partial g}{\partial x}-\frac{\partial f}{\partial y}\right)}_{\text{2D curl}} dA \text{ (Green's Theorem)}$$

$$\text{3D: circulation} = \oint_C \vec{F}\cdot d\vec{r} = \iint_S \underbrace{\left(\nabla\times\vec{F}\right)}_{\text{3D curl}}\cdot\vec{n}\,dS \text{ (Stokes' Theorem)}$$

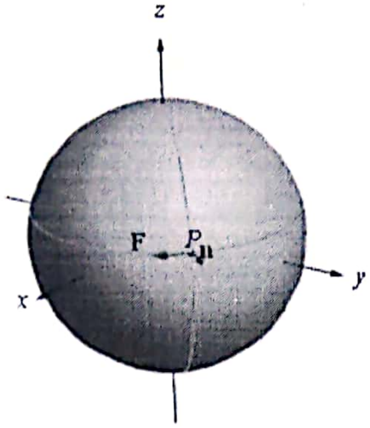
- The flux form of Green's Theorem → The Divergence Theorem

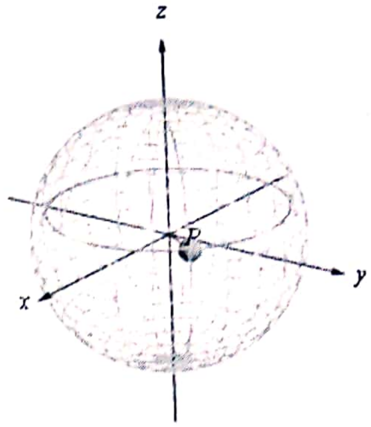
$$\text{2D: flux} = \oint_C \vec{F}\cdot\vec{n}\,ds = \iint_R \underbrace{\left(\frac{\partial f}{\partial x}+\frac{\partial g}{\partial y}\right)}_{\text{2D divergence}} dA \text{ (Green's Theorem)}$$

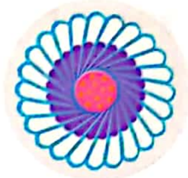
$$\text{3D: flux} = \iint_S \vec{F}\cdot\vec{n}\,dS = \iiint_D \underbrace{\left(\nabla\cdot\vec{F}\right)}_{\text{3D Divergence}} dV \text{ (Divergence Theorem)}$$

The line integral on the left gives the flux across the boundary of R. The double integral on the right measures the net expansion or contraction of the vector field within R. If $\vec{F}$ represents a fluid flow or the transport of a material, the theorem says that the cumulative effect of the sources (or sinks) of the flow within R equals the net flow across its boundary.

The Divergence Theorem is a direct extension of Green's Theorem. The plane region in Green's Theorem becomes a solid region D in $\mathbb{R}^3$, and the closed curve in Green's Theorem becomes the oriented surface S that encloses D. The flux integral in Green's Theorem becomes a surface integral over S, and the double integral in Green's Theorem becomes a triple integral over D of the three-dimensional divergence.







check out Manipulate 17,68

Theorem: Divergence Theorem (also called Gauss' Theorem)

Let $\vec{F}$ be a vector field whose components have continuous first partial derivatives in a connected and simply connected region *D* enclosed by a smooth oriented surface *S*. Then

$$\text{flux} = \iint_S \vec{F}\cdot\vec{n}\,dS = \iiint_D \underbrace{\left(\nabla\cdot\vec{F}\right)}_{3D\,\text{Divergence}}\,dV$$

Where $\vec{n}$ is the outward unit normal vector on *S*.

Ex1: Let S be the sphere $x^2+y^2+z^2=a^2$ oriented outward. Use the Divergence Theorem to find the flux of the vector field $\vec{F}(x,y,z)=z\vec{k}$ across S.

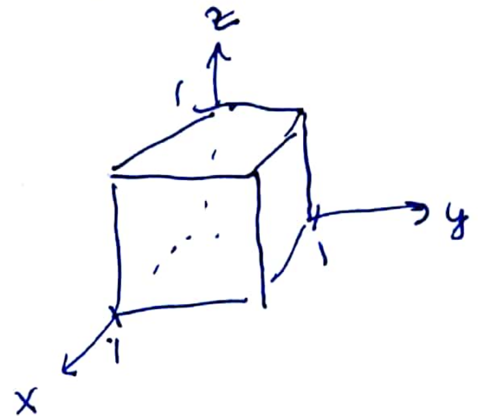
$$\text{flux} = \iint_S \vec{F}\cdot\vec{n}\,dS$$

$$= \iiint_{\text{sphere}} \operatorname{div}(\vec{F})\,dV$$

$$= \iiint_{\text{sphere}} 1\,dV$$

$$= \frac{4\pi a^3}{3}.$$

Ex2: Let S be the cube that is placed on the first octant at the origin with each side of length 1, oriented outward. Find the flux of the radial vector field $\vec{F}(x,y,z)=2x\vec{i}+3y\vec{j}+z^2\vec{k}$ across S.



$$\text{flux} = \iint_{\substack{\text{sides}\\ \text{of}\\ \text{box}}} \vec{F}\cdot\vec{n}\, dS$$

$$= \iiint_{\substack{\text{interior}\\ \text{of}\\ \text{box}}} (2+3+2z)\, dV$$

$$= \int_0^1\int_0^1\int_0^1 (5+2z)\, dx\, dy\, dz$$

$$= 5 + \int_0^1 2z\, dz$$

$$= 6$$

Note: With Stokes' Theorem, rotation fields are noteworthy because they have a nonzero curl. With the Divergence Theorem, the situation is reversed. Pure rotation fields have zero divergence. However, with the Divergence Theorem, radial fields are interesting and have many physical applications.

Ex3: Let S be the surface of the solid enclosed by the circular cylinder $x^2+y^2=9$ and the planes $z=0$ and $z=2$, oriented outward. Use the Divergence Theorem to find the flux of the vector field $\vec{F}(x,y,z)=x^3\vec{i}+y^3\vec{j}+z^2\vec{k}$ across S.

$$\text{flux} = \iint_S \vec{F}\cdot\vec{n}\,dS$$

$$= \iiint_{\text{cylinder}} (3x^2+3y^2+2z)\,dV$$

$$= \int_0^2\int_0^{2\pi}\int_0^3 (3r^2+2z)\,r\,dr\,d\theta\,dz$$

$$= 2\pi\int_0^2\int_0^3 3r^3+2zr\,dr\,dz$$

$$= 2\pi\int_0^2 \left[\frac{3}{4}r^4+zr^2\right]_0^3 dz$$

$$= 2\pi\int_0^2 \left(\frac{243}{4}+9z\right)dz$$

$$= 2\pi\left[\frac{243}{4}z+\frac{9}{2}z^2\right]_0^2$$

$$= \pi(243+36)$$

$$= 279\pi$$

Interpreting the Divergence Theorem: Suppose $\vec{v}$ is the velocity field of a material, such as water or molasses, and ρ is its constant density. The vector field $\vec{F}=\rho\vec{v}$ describes the mass transport of the material. This means that $\vec{F}$ gives the mass of material flowing past a point (in each of the three coordinate directions) per unit of surface area per unit of time. When $\vec{F}$ is multiplied by an area, the result is the flux.

The Divergence Theorem shows that the net flux (mass being transported) across the boundary surface is equal to the net divergence across the region inside the surface.

More succinctly, the flux of material (fluid, heat, electric field lines) across the boundary of a region is the cumulative effect of the sources within the region.

Ex4: Let S be the surface of the solid enclosed by the hemisphere $z=\sqrt{a^2-x^2-y^2}$ and the plane $z=0$, oriented outward. Use the Divergence Theorem to find the flux of the vector field $\vec{F}(x,y,z)=x^3\vec{i}+y^3\vec{j}+z^3\vec{k}$ across S.

$$\text{flux} = \iint\limits_{\text{surface}} \vec{F}\cdot\vec{n}\, ds$$

$$= \iiint\limits_{\text{solid}} 3(x^2+y^2+z^2)\, dV$$

$$= \int_0^{\frac{\pi}{2}}\int_0^{2\pi}\int_0^{a} 3\rho^2\cdot\rho^2\sin\phi\, d\rho\, d\theta\, d\phi$$

$$= 3\left[\frac{\rho^5}{5}\right]_0^{a}\left[\theta\right]_0^{2\pi}\left[-\cos\theta\right]_0^{\pi/2}$$

$$= 3\cdot\frac{a^5}{5}\cdot 2\pi\cdot 1$$

$$= \frac{6}{5}\pi a^5$$

We have reached the conclusion of calculus (at least until you seek out a more advanced course as part of your studies to be a mathematics major). To help understand what has been accomplished, consider the following lengthy adaptation t from George Simmon's calculus text titled "Maxwell's Equations, A Final Thought."

To gain a slight glimpse of the significance of the ideas of this chapter, we look very briefly at the famous equations formulated in the 1860's by James Clerk Maxwell (1831-1879). These equations are remarkable because they contain a complete theory of everything that was then known or would later become known about electricity and magnetism.

In Maxwell's theory there are two vector fields defined at every point in space: an electric field $\vec{E}$ and a magnetic field $\vec{B}$. The electric field is produced by charged particles (electrons, protons, etc.) that may be moving or stationary, and the magnetic field by moving charged particles.

All known phenomena involving electromagnetism can be explained and understood by means of Maxwell's Equations which written in the notation of this course are:

- $\iint\limits_S \vec{E}\cdot\vec{n}\,dA = \dfrac{Q}{\varepsilon_0}$
  - Meaning: flux of $\vec{E}$ through a closed surface = $\dfrac{\text{charge inside}}{\varepsilon_0}$
- $\oint\limits_C \vec{E}\cdot d\vec{r} = -\dfrac{\partial}{\partial t}\iint\limits_S \vec{B}\cdot\vec{n}\,dA$
  - Meaning: line integral of $\vec{E}$ around a loop = $-\dfrac{\partial}{\partial t}\left(\text{flux of } \vec{B} \text{ thru the loop}\right)$
- $\iint\limits_S \vec{B}\cdot\vec{n}\,dA = 0$
  - Meaning: flux of $\vec{B}$ through a closed surface = 0
- $c^2\oint\limits_C \vec{B}\cdot d\vec{r} = \dfrac{1}{\varepsilon_0}\iint\limits_S \vec{j}\cdot\vec{n}\,dA + \dfrac{\partial}{\partial t}\iint\limits_S \vec{E}\cdot\vec{n}\,dA$
  - Meaning: c^2 (integral of $\vec{B}$ around a loop) = $\dfrac{\text{current thru loop}}{\varepsilon_0} + \dfrac{\partial}{\partial t}\left(\text{flux of } \vec{E} \text{ thru loop}\right)$

Here Q relates to charge, ε_0 is a constant, c is the speed of light, and $\vec{j}$ is the current density (not to be confused with the unit vector in the direction of the y-axis). We make no attempt to discuss the meaning of these four equations, be we do point out that the first two make statements about the divergence and curl of $\vec{E}$, and the second two about the divergence and curl of $\vec{B}$.

Our only purpose in mentioning these matters is to try to make it perfectly clear to the student that the mathematics we have been doing in this chapter has profoundly important applications in physical science. The Nobel Prize winning physicist Richard Feynman devotes the first 21 chapters in vol. 2 of his *Lectures* to the meaning and implications of Maxwell's equations. At one point he memorably remarks:

> From a long view of the history of mankind – seen from, say, ten thousand years from now – there can be little doubt that the most significant event of the 19th century will be judged as Maxwell's discovery of the laws of electrodynamics. The American Civil War will pale into provincial insignificance in comparison with this important scientific event of the same decade.

In making this provocative comment, perhaps Feynman was carried away by his ebullient enthusiasm – but perhaps not.

THE END! PTL!