HW: 25 min + 7Q's



Green's Theorem set the stage for the final act in our exploration of calculus. The remainder of this course makes this our goal: to lift both forms of Green's Theorem out of the plane ($\mathbb{R}^2$) and into space ($\mathbb{R}^3$). It is done as follows:

- The circulation form of Green's Theorem relates a line integral over a simple closed oriented curve in the plane to a double integral over the enclosed region. In an analogous manner, we will see that Stokes' Theorem (Section 17.7) relates a line integral over a simple closed oriented curve in $\mathbb{R}^3$ to a double integral over a surface whose boundary is the same curve.
- The flux form of Green's Theorem relates a line integral over a simple closed oriented curve in the plane to a double integral over the enclosed region. Similarly, the Divergence Theorem (Section 17.8) relates an integral over a closed oriented surface in $\mathbb{R}^3$ to a triple integral over the region enclosed by that surface.

In order to make these extensions, we need a few more tools.

- The two-dimensional divergence and two-dimensional curl must be extended to three dimensions (this section). Applications include fluid flow and electricity and magnetism.
- The idea of a surface integral must be introduced (the next topic).

## Divergence and Curl

Objective:

1. Calculating divergence and curl
2. Examples/visualizations

### *I. Calculating divergence and curl*

Divergence and curl are relatively easy to calculate if you make use of the del operator:

$$\nabla = \vec{i}\,\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z}$$

Using the del operator, we can easily calculate the gradient, divergence, and curl

- Gradient: $\nabla f = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$
- Divergence: $\nabla \cdot \vec{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle f, g, h \rangle = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$
- Curl: $\nabla \times \vec{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \times \langle f, g, h \rangle = \left\langle \left( \frac{\partial h}{\partial y} - \frac{\partial g}{\partial z} \right), \left( \frac{\partial f}{\partial z} - \frac{\partial h}{\partial x} \right), \left( \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) \right\rangle$

**Ex1**: If $\vec{F}(x,y,z) = xz\vec{i} + xyz\vec{j} - y^2\vec{k}$, find the divergence and curl of $\vec{F}$. Is $\vec{F}$ conservative?

Divergence

$$\nabla \cdot \vec{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle xz, xyz, -y^2 \rangle$$

$$= z + xz + 0.$$

curl

$$\nabla \times \vec{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \times \langle xz, xyz, -y^2 \rangle$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & xyz & -y^2 \end{vmatrix}$$

$$= \langle -2y - xy, -(0-0), yz - x \rangle$$

$$h_y - g_z \neq 0 \iff h_y \neq g_z$$

$\therefore \vec{F}$ is not conservative.

Theorem: If $f$ is a function of three variables that has continuous second-order partial derivatives, then $\operatorname{curl}(\nabla f) = 0$

Since a conservative vector field is one for which $\vec{F} = \nabla f$, then we can rephrase the above theorem as follows: If $\vec{F}$ is conservative, then $\operatorname{curl}\vec{F} = \vec{0}$

Theorem: If $\vec{F}$ is a vector field defined on all of $\mathbb{R}^3$ whose component functions have continuous partial derivatives and $\operatorname{curl}\vec{F} = \vec{0}$, then $\vec{F}$ is a conservative vector field.

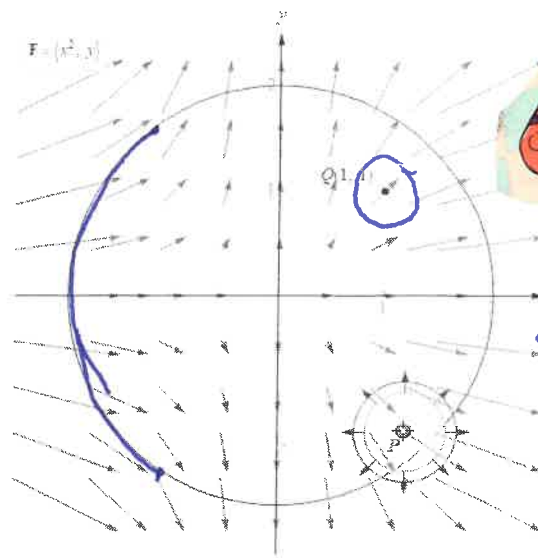
## 2. Examples/visualizations

Ex 2: To gain some intuition about the divergence, consider the two-dimensional vector field $\vec{F} = \langle x^2, y \rangle$ and a circle $C$ of radius 2 centered at the origin.

- Without computing it, determine whether the two-dimensional divergence is positive or negative at the point Q(1,1). Why?

  more in than out.

  div > 0



- Confirm your conjecture in part (a) by computing the two-dimensional divergence at Q.

  $\operatorname{div} \vec{F} = 2x + 1 \Big|_{(1,1)} \quad 3$

check out manipulate 17.39

- Based on part (b), over what regions within the circle is the divergence positive and over what regions within the circle is the divergence negative?

  $2x + 1 > 0 \Rightarrow 2x > -1 \Rightarrow x > -\frac{1}{2}$ (div > 0)

  and div < 0 on $x < -\frac{1}{2}$.

- By inspection of the figure, on what part of the circle is the flux across the boundary outward? Is the net flux out of the circle positive or negative?

  flux appears negative (inward) when $x < -1$.

  and positive (outward) $x > -1$. The net flux appears positive

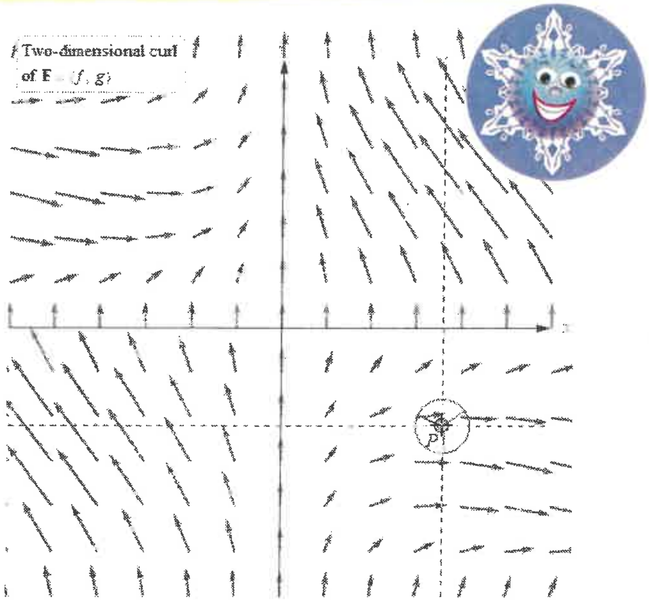
The reason for the name *divergence* can be understood in the context of fluid flow. If $\vec{F}$ is the velocity of a fluid (or gas), then $\operatorname{div}(\vec{F})$ represents the net rate of change (with respect to time) of the mass of fluid (or gas) flowing from the point $(x,y,z)$ per unit volume. In other words, $\operatorname{div}(\vec{F})$ measures the tendency of the fluid to diverge from the point.

Note: If $\operatorname{div}(\vec{F}) = 0$, then $\vec{F}$ is said to be incompressible.

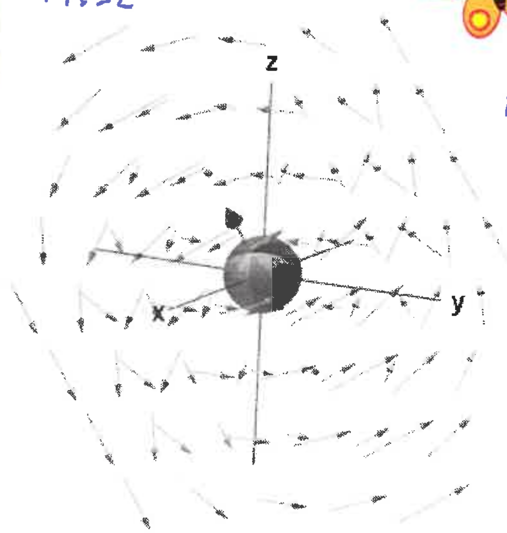
Visualizing and building intuition about curl is challenging as it involves vector fields, three dimensions, and motion … none of which are easy to describe on a static page.

The website mathinsight.org has some excellent visuals. They write the vector field $\vec{F}$ determines both in what direction the sphere rotates, and the speed at which it rotates. We define the curl of $\vec{F}$, denoted $\text{curl}\left(\vec{F}\right)$, by a vector that points along the axis of the rotation and whose length corresponds to the speed of the rotation.

Check out manipulate 17.32




check out the website mathinsight.org/curl_idea




This macroscopic circulation of fluid around circles (i.e., the rotation you can easily view in the above graph) actually is not what curl measures. Curl is more like "microscopic circulation." To test for curl, imagine that you immerse a small sphere into the fluid flow, and you fix the center of the sphere at some point so that the sphere cannot follow the fluid around. Although you fix the center of the sphere, you allow the sphere to rotate in any direction around its center point. The rotation of the sphere measures the curl of the vector field $\vec{F}$ at the point in the center of the sphere. (The sphere should actually be really really small, because, remember, the curl is microscopic circulation.)[1]

Note: If $\text{curl}\left(\vec{F}\right)=\vec{0}$, then $\vec{F}$ is said to be irrotational.

Theorem: If $\vec{F}=\langle f,g,h\rangle$ is a vector field on $\mathbb{R}^3$ and $f,g,h$ have continuous second-order partial derivatives then $\text{div}\left(\text{curl}\left(\vec{F}\right)\right)=0$

[1] https://mathinsight.org/curl_idea