HW: 55 min + 7 Q's

Conservative Vector Fields

This is an action-packed section in which several fundamental ideas come together. At the heart of the matter are two questions.

1. When can a vector field be expressed as the gradient of a potential function? A vector field with this property will be defined as a conservative vector field.
2. What special properties do conservative vector fields have?

In this section we present a test to determine whether a vector field in $\mathbb{R}^2$ or $\mathbb{R}^3$ is conservative. The test is followed by a procedure to find a potential function for a conservative field. We then develop several equivalent properties shared by all conservative vector fields.

I. Conservative Fields

Recall that given a function $f(x,y)$, we can find the corresponding gradient field ∇f. The following definition reverses this relationship.

Definition: Conservative Vector Field

A vector field $\vec{F}$ is said to be conservative if there exists a scalar function φ such that $\vec{F} = \nabla\varphi$.

There are a number of important conservative fields in physics which prompts us to want to understand them better. As you may recall from ~~Clauret's~~ Clairaut's Theorem, if a function has continuous second partial derivatives, the order of differentiation in the second partial derivatives does not matter ($f_{xy} = f_{yx}$). This leads us to the following test.

Method: Test for conservative vector fields (easy):

Let $\vec{F} = \langle f, g, h\rangle$ be a vector field defined on a connected and simply connected region *D* of $\mathbb{R}^3$, where *f*, *g*, and *h* have continuous first partial derivatives on *D*. Then, $\vec{F}$ is a conservative vector field on *D* (there is a potential function φ such that $\vec{F} = \nabla\varphi$) if and only if

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}, \quad \frac{\partial f}{\partial z} = \frac{\partial h}{\partial x}, \quad \frac{\partial g}{\partial z} = \frac{\partial h}{\partial y}.$$

For vector fields in $\mathbb{R}^2$, we have the single condition $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$.

Ex1: Determine if $\vec{F} = \langle \underbrace{3y+4}_{f}, \underbrace{3x+5}_{g} \rangle$ is a conservative vector field. If it is conservative, find its potential function.

Test: $g_x = 3 = f_y$. Yes $\vec{F}$ is conservative

Find φ

(1.) Integrate $\varphi_x = f$ WRT x to obtain φ

$\varphi = \int (3y+4)\,dx = 3xy + 4x + c(y)$ constant WRT x

(2.) Differentiate and solve $\varphi_y = g$ to find $c'(y)$

$\frac{\partial}{\partial y}(3xy + 4x + c(y)) = 3x + c'(y) = 3x+5 \Rightarrow c'(y) = 5$

(3). Integrate $c'(y)$ WRT y to obtain $c(y)$.

$c(y) = \int 5\,dy = 5y + k$

$\therefore \varphi(x,y) = 3xy + 4x + 5y + k$.

Like antiderivatives, potential functions are determined up to an arbitrary additive constant. Unless an additive constant in a potential function has some physical meaning, it is usually omitted. Given a conservative vector field, there are several methods for finding a potential function.

Procedure: Finding Potential Functions in $\mathbb{R}^3$

Suppose $\vec{F} = \langle f, g, h \rangle$ is a conservative vector field. To find φ such that $\vec{F} = \nabla\varphi$, use the following steps:

1. Integrate $\varphi_x = f$ with respect to x to obtain φ, which includes an arbitrary function $c(y,z)$.
2. Differentiate to compute φ_y and equate it to g to obtain an expression for $c_y(y,z)$.
3. Integrate $c_y(y,z)$ with respect to y to obtain $c(y,z)$, including an arbitrary function $d(z)$.
4. Differentiate to compute φ_z and equate it to h to get $d'(z)$.
5. Integrate $d'(z)$ with respect to z to obtain $d(z)$.

Beginning the procedure with $\varphi_y = g$ or $\varphi_z = h$ may be easier in some cases.

Ex2: Determine if $\bar{G}=\left\langle \underbrace{yz+2y+3z+5}_{f},\ \underbrace{xz+2x+4z+6}_{g},\ \underbrace{xy+3x+4y+7}_{h}\right\rangle$ is a conservative vector field. If it is conservative, find its potential function.

Test: $f_y = z+2 = g_x$; $f_z = y+3 = h_x$; $g_z = x+4 = h_y$

so $\vec{G}$ is conservative.

Find φ:

(1) Integrate f WRT x to find φ.

$$\varphi = \int (yz+2y+3z+5)\,dx = xyz + 2xy + 3xz + 5x + c(y,z)$$

constant WRT x

(2) solve $\varphi_y = g$ to find $c_y(y,z)$

$$\varphi_y = xz + 2x + c_y(y,z) = xz + 2x + 4z + 6$$

$$\Rightarrow c_y(y,z) = 4z + 6$$

(3) Integrate $c_y(y,z)$ WRT y to find $c(y,z)$

$$c(y,z) = \int (4z+6)\,dy = 4yz + 6y + d(z)$$

constant WRT z

(4) solve $\varphi_z = h$ to find $d'(z)$

$$\varphi_z = xy + 3x + 4y + d'(z) = xy + 3x + 4y + 7$$

$$\Rightarrow d'(z) = 7$$

(5) Integrate $d'(z)$ WRT to z to find $d(z)$

$$d(z) = \int 7\,dz = 7z + k$$

$$\therefore \varphi = xyz + 2xy + 3xz + 5x + 4yz + 6y + 7z$$

II. Conservative vector fields and line integrals

In general, when calculating a line integral, we must begin by parameterizing the path because different paths lead to different amounts of work (to focus on a particular application). Not so if the field is conservative.

Theorem: Fundamental Theorem for Line Integrals

> Let R be a region of $\mathbb{R}^2$ or $\mathbb{R}^3$ and let φ be a differentiable potential function defined on R. If $\bar{F} = \nabla\varphi$ (which means that $\bar{F}$ is conservative), then
>
> $$\int_C \bar{F}\cdot\bar{T}\,ds = \int_C \bar{F}\cdot d\bar{r} = \varphi(B) - \varphi(A)$$
>
> for all points A and B in R and all piecewise-smooth oriented curves C in R from A to B.

Here is the meaning of the Fundamental Theorem of Line Integrals: If $\bar{F}$ is a conservative vector field, then the value of a line integral of $\bar{F}$ depends only on the endpoints of the path. For this reason, we say the line integral is independent of path, which means that to evaluate line integrals of conservative vector fields, we do not need a parameterization of the path.

The proof is in the text and is notable for one important detail: It makes use of the multivariate chain rule.

$$\frac{d\varphi}{dt} = \frac{\partial\varphi}{\partial x}\frac{dx}{dt} + \frac{\partial\varphi}{\partial y}\frac{dy}{dt} + \frac{\partial\varphi}{\partial z}\frac{dz}{dt}$$

Ex1 revisited: We previously showed $\bar{F} = \left\langle \underbrace{3y+4}_{f}, \underbrace{3x+5}_{g} \right\rangle$ is a conservative vector field and found its potential function. Find the work to travel through this field around the unit circle from $\overset{A}{(1,0)}$ to $\overset{B}{(-1,0)}$

recall: $\varphi(x,y) = 3xy + 4x + 5y$

work $= \int_C \vec{F}\cdot d\vec{r} = \int_C \nabla\varphi\cdot d\vec{r} = \varphi(B) - \varphi(A) = -4-4$

conservative

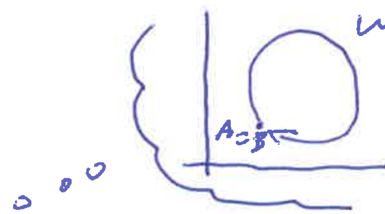
$\therefore$ The work is -8 units.

Ex2 revisited: We previously showed $\vec{G} = \left\langle \underbrace{yz+2y+3z+5}_{f},\ \underbrace{xz+2x+4z+6}_{g},\ \underbrace{xy+3x+4y+7}_{h} \right\rangle$ is a conservative vector field and found its potential function. Find the work to travel through this field along the twisted cubic from (0,0,0) to (1,1,1).

recall: $\varphi(x,y,z) = xyz + 2xy + 3xz + 4yz + 5x + 6y + 7z$

$$\text{Work} = \int_C \vec{F}\cdot d\vec{r} = \int_C \nabla\varphi \cdot d\vec{r} = \varphi(B) - \varphi(A) = 28$$

$\therefore$ The work is 28 units.



work $= \varphi(B) - \varphi(A)$
$= 0.$

Question: What is the work to travel around a *closed path* through a *conservative vector field?*

Theorem: Line Integrals on Closed Curves

Let R be an open connected region in $\mathbb{R}^2$ or $\mathbb{R}^3$. Then $\vec{F}$ is a conservative vector field on R if and only if $\oint_C \vec{F}\cdot d\vec{r} = 0$ on all simple closed piecewise-smooth oriented curves C in R.

There are a variety of related claims and theorems that can be made by adjusting hypothesis. At the same time, the ones included here are at the heart of the course and others can be picked up as they are needed.

Summary: Properties of conservative vector fields

We have established three equivalent properties of conservative vector fields $\vec{F}$ defined on an open connected region R in $\mathbb{R}^2$ (or D in $\mathbb{R}^3$).

- There exists a potential function φ such that $\vec{F} = \nabla\varphi$ (definition).
- $\int_C \vec{F}\cdot d\vec{r} = \varphi(B) - \varphi(A)$ for all points A and B in R and all piecewise-smooth oriented curves C from A to B (path independence).
- $\oint_C \vec{F}\cdot d\vec{r} = 0$ on all simple piecewise-smooth closed oriented curves C in R.

These can be stated succinctly as $\boxed{\text{path independence}} \Leftrightarrow \boxed{\vec{F}\text{ conservative}} \Leftrightarrow \boxed{\oint_C \vec{F}\cdot d\vec{r} = 0}$

Last but not least, those who have studied physics are probably wondering the connection between conservative and potential as outlined in calculus and conservative forces and potential energy as outlined in physics …

Ex3: How much work is required to move a particle along a curve $\vec{r}(t)$, $a \le t \le b$ (we will call the endpoints A and B). step 1: kinetic energy.

Formulas we will need:

- Newton's second law: $\vec{F} = m\vec{a}$
- Kinetic energy: $KE = \frac{1}{2}mv^2$
- Potential energy: $PE = -\varphi$ (why?)

$$\text{work} = \int_C \vec{F} \cdot d\vec{r}$$ (Do not assume $\vec{F}$ is conservative).

$$= \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt$$

$$= \int_a^b m\,\vec{r}\,''(t) \cdot \vec{r}\,'(t)\,dt$$

$$= m\int_a^b \frac{1}{2}\frac{d}{dt}\left[\vec{r}\,'(t) \cdot \vec{r}\,'(t)\right]dt$$ ... product rule in reverse

$$= \frac{m}{2}\int_a^b \frac{d}{dt}\left|\vec{r}\,'(t)\right|^2 dt$$

$$= \frac{m}{2}\Big[\left|\vec{r}\,'(t)\right|^2\Big]_a^b$$

$$= \frac{m}{2}\left[\left|\vec{r}\,'(b)\right|^2 - \left|\vec{r}\,'(a)\right|^2\right]$$ recall $\vec{r}\,' = \vec{v}$

$$= \frac{m}{2}\left[\left|\vec{v}(b)\right|^2 - \left|\vec{v}(a)\right|^2\right]$$ the velocity at points A & B.

$$= KE(B) - KE(A)$$

step 2: potential energy

suppose $\vec{F}$ is conservative: $\vec{F} = \nabla\varphi$ for some φ

In physics, we define PE: $PE = -\varphi$ (why?)

$\Rightarrow \nabla PE = -\nabla\varphi$ (this should make intuitive sense) so $\text{work} = \int_C \vec{F} \cdot d\vec{r}$

$= \int_C \nabla \varphi \cdot d\vec{r}$

$= - \int_C \nabla PE \cdot d\vec{r}$

$= - \left(PE(\vec{r}(b)) - PE(\vec{r}(a)) \right)$

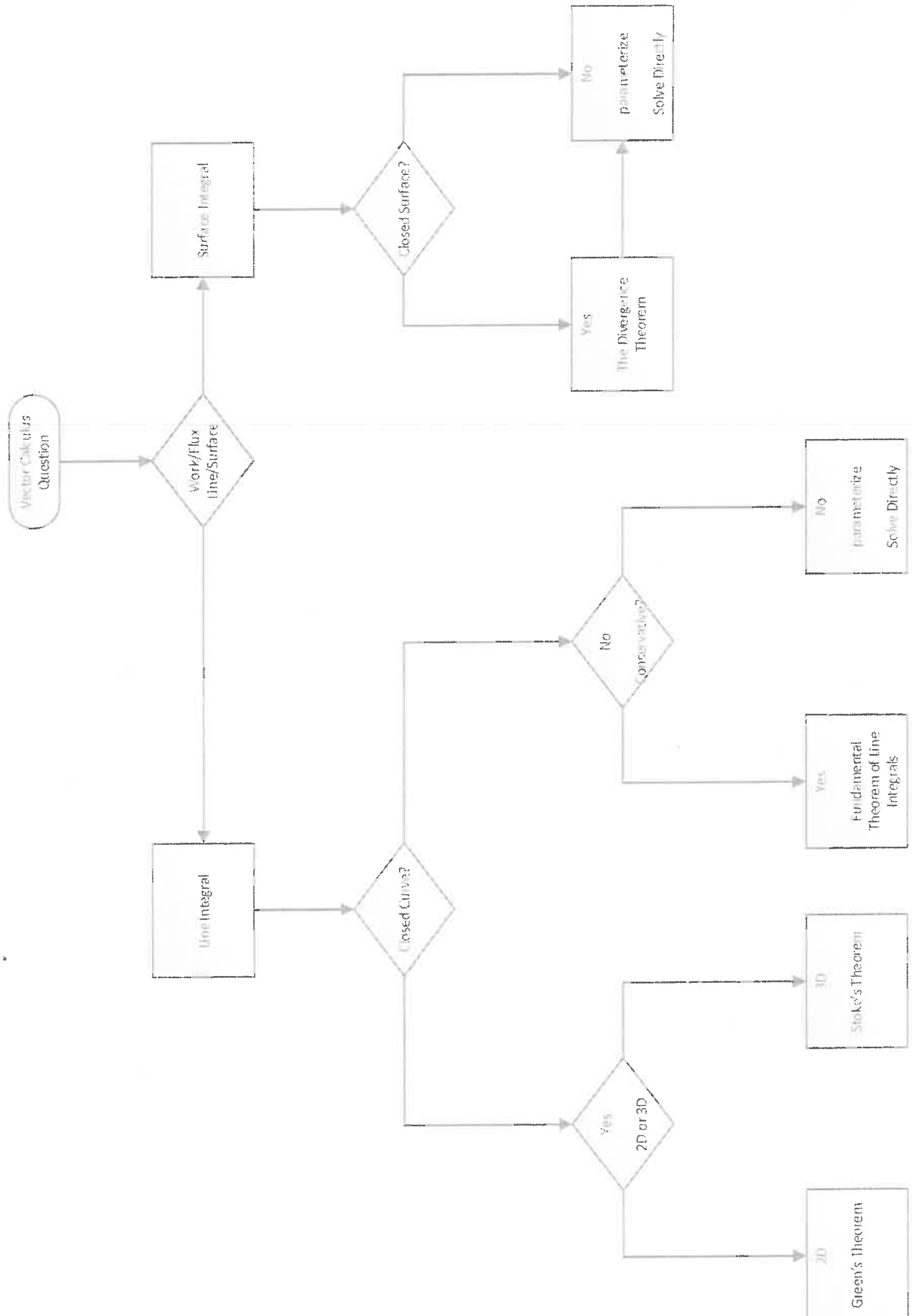
$= PE(A) - PE(B)$

Step 3: The conclusion.

$\text{work} = KE(B) - KE(A) = PE(A) - PE(B)$

$\Rightarrow PE(A) + KE(A) = PE(B) + KE(B)$

That is, energy is conserved :).



Vector Calculus Question
Work/Flux
Line/Surface
Surface Integral
Closed Surface?
Yes
The Divergence Theorem
No
parameterize
Solve Directly
Line Integral
Closed Curve?
No
Conservative?
Yes
Fundamental Theorem of Line Integrals
No
parameterize
Solve Directly
Yes
2D or 3D
3D
Stoke's Theorem
2D
Green's Theorem