Change of Variables in Multiple Integrals

Objective:

1. Change of variable in a single integral - u substitution
2. Where might this have previously helped and examples
3. The Theory
4. Examples continued

In this section we will discuss a general method of evaluating double and triple integrals by substitution. Most of the results in this section are very difficult to prove, so our approach will be informal and motivational. Our goal is to provide a geometric understanding of the basic principles and an exposure to computational techniques.

1. Change of Variable in a Single Integral

$I =$

Ex0: $\displaystyle\int_a^b \sin(2x)\,dx$

$$\text{Let } u = 2x$$

$$du = 2\,dx$$

↑ scaling factor

↓

$$I = \int_{2a}^{2b} \sin(u) \cdot \frac{1}{2}\,du$$

↑ limits change

↑ integrand changes.

$y = \sin 2x$

$\dfrac{\pi}{2}$

$y = \sin x$

π
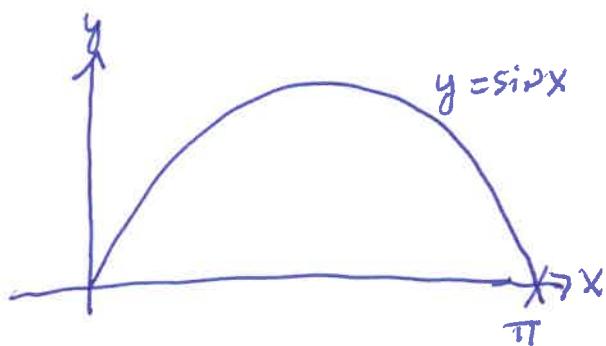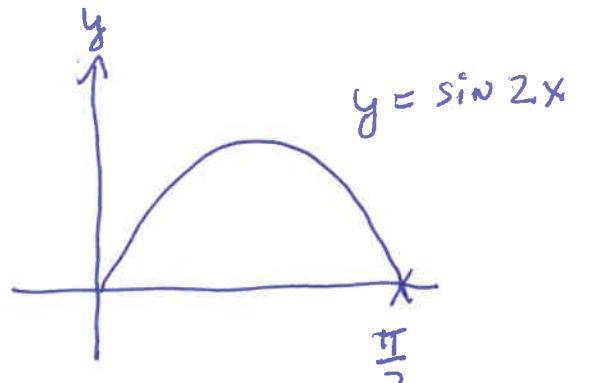
The main point: Substitution requires changing the integrand, limits of integration, and the introduction of a scaling factor.

2. Where might this have previously helped and examples

Ex1: Use a triple integral to compute the volume of the wedge of the square column $|x|+|y|=10$ created by the planes $z=0$ and $x+y+z=10$.
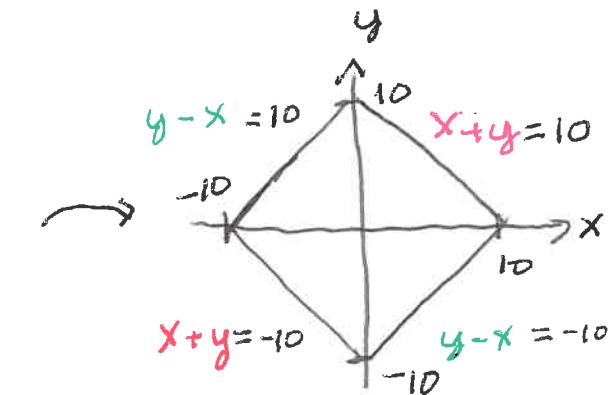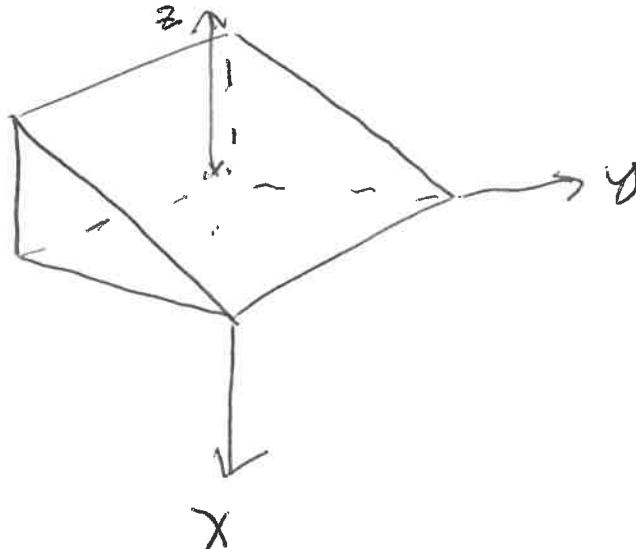
Let $u = x+y$ & $v = y-x$
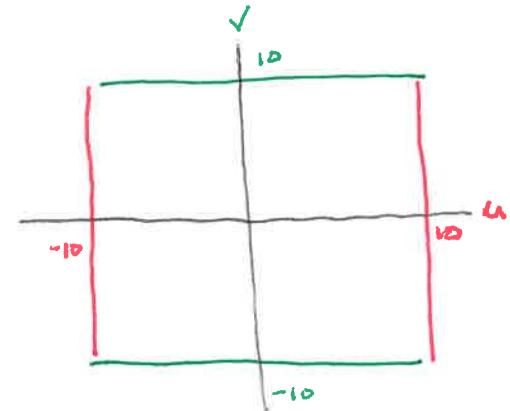
$$V = \int_{-10}^{10}\int_{-10}^{10}\int_{0}^{10-u} 1 \cdot \left|\,^{scaling}_{factor}\,\right|\, dz\, du\, dv$$

$$\uparrow$$
$$\frac{1}{2}$$

$$= \int_{-10}^{10}\int_{-10}^{10} (10-u)\cdot\left|\frac{1}{2}\right|\, du\, dv$$

$$= \frac{1}{2}\cdot 20\left[10u - \frac{u^2}{2}\right]_{-10}^{10}$$

$$= 10\left[(100-50)-(-100-50)\right]$$

$$= 2000.$$

Area $(x,y) = 200$

Area $(u,v) = 400$

You can find the scaling factor both ways

① Area $(x,y) = \frac{1}{2}$ Area (u,v)

② Area $(u,v) = 2$ Area (x,y)

Everything but the scaling factor.

Ex2: Set up an iterated integral to find

$$I = \iint\limits_D \sin(9x^2 + 4y^2)\, dA \quad \text{where } D \text{ is the}$$

region in Q1 bounded by $9x^2 + 4y^2 = 1$

Solution: Let $u = 3x$ and $v = 2y$

$$\Rightarrow I = \iint\limits_{\substack{u^2+v^2 \le 1 \\ \text{and in } Q_1}} \sin(u^2 + v^2) \left|\begin{array}{c}\text{scaling} \\ \text{factor}\end{array}\right| dA'$$

↖ in terms of (u, v)

Ex3: set up an iterated integral to find

$$I = \iint\limits_D \frac{x - 2y}{3x - y}\, dA \quad \text{where } D \text{ is the parallelogram}$$

enclosed by $x - 2y = 0$; $x - 2y = 4$; $3x - y = 1$; $3x - y = 8$

Solution: Let $u = 3x - y$ and $v = x - 2y$

$$\Rightarrow I = \int_0^4 \int_1^8 \frac{v}{u} \left|\begin{array}{c}\text{scaling} \\ \text{factor}\end{array}\right| du\, dv$$

In order for a multivariate substitution to be helpful, the substitution must "help" both integrand AND limits of integration.

Ex 4: Set up an iterated integral to represent

$$I = \iint_D (x+y) e^{x^2-y^2} \, dA \quad \text{where } D \text{ is the region}$$

enclosed by $x+y=0$; $x+y=3$; $x-y=0$; $x-y=2$

solution: Let $u = x+y$ and $v = x-y$

$$\Rightarrow I = \int_0^2 \int_0^3 u e^{uv} \left| \text{scaling factor} \right| \, du \, dv$$

The scaling factor = |The Jacobian.|

ex1 rev.

recall $u = x + y$

$v = y - x$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = 2$$

Area $(u,v) = 2$ Area (x,y)

if you solve for x and y.

$$x = \frac{1}{2}u - \frac{1}{2}v$$

$$y = \frac{1}{2}u + \frac{1}{2}v$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{2}$$

Area$(x,y) = \frac{1}{2}$ Area (u,v)

3. The Theory

To derive the change-of-variable formula for double integrals, we need to understand the relationship between the area of a small rectangular region in the uv-plane and the area of its image in the xy-plane under a transformation T given by equations $x = x(u,v)$ and $y = y(u,v)$. For this purpose, suppose that Δu and Δv are positive and small, and consider a rectangular region S in the uv-plane enclosed by the lines

$$u = u_0, \quad u = u_0 + \Delta u, \quad v = v_0, \quad v = v_0 + \Delta v$$

Then R in xy-plane is its image. Let

$$\vec{r}(u,v) = \langle x(u,v), y(u,v) \rangle$$
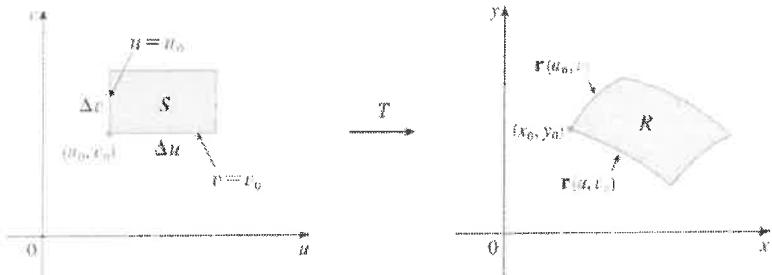
be the position vector to the point in xy-plane that corresponds to the point (u,v) in the uv-plane, then

$$\vec{r}(u, v_0) = \langle x(u,v_0), y(u,v_0) \rangle \text{ corresponds to } v = v_0$$

and

$$\vec{r}(u_0, v) = \langle x(u_0,v), y(u_0,v) \rangle \text{ corresponds to } u = u_0.$$

Since we assumed that Δu and Δv are small, the region R can be approximated by a parallelogram determined by the secant vectors:

$$\vec{a} = \vec{r}(u_0 + \Delta u, v_0) - \vec{r}(u_0, v_0)$$

$$\vec{b} = \vec{r}(u_0, v_0 + \Delta v) - \vec{r}(u_0, v_0)$$
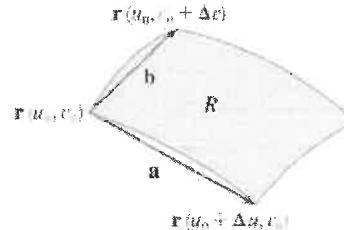
A more useful approximation of R can be obtained by using tangent vectors as follows:

$$\vec{a} = \frac{\vec{r}(u_0 + \Delta u, v_0) - \vec{r}(u_0, v_0)}{\Delta u} \Delta u \approx \frac{\partial \vec{r}}{\partial u} \Delta u = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u} \right\rangle \Delta u = \vec{r}_u \Delta u$$
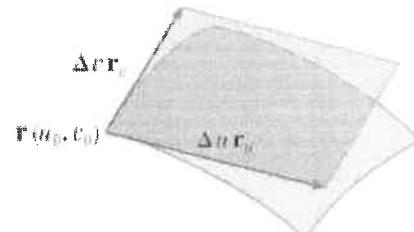
$$\vec{b} = \frac{\vec{r}(u_0, v_0 + \Delta v) - \vec{r}(u_0, v_0)}{\Delta v} \Delta v \approx \frac{\partial \vec{r}}{\partial v} \Delta v = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v} \right\rangle \Delta v = \vec{r}_v \Delta v$$

where the partial derivatives are evaluated at (u_0, v_0).

At $T(u_0, v_0) = (x_0, y_0)$ the vectors $\vec{r_u}$ and $\vec{r_v}$ are tangent to $\vec{r}(u, v_0)$ and $\vec{r}(u_0, v)$ respectively, and so are $\vec{r_u}\Delta u$ and $\vec{r_v}\Delta v$. Thus the area of R which we denote by ΔA can be approximated by the area of the parallelogram determined by these vectors.

$$\Delta A \approx \left| \vec{r_u}\Delta u \times \vec{r_v}\Delta v \right| = \left| \vec{r_u} \times \vec{r_v} \right| \Delta u \Delta v$$

where the partial derivatives are evaluated at (u_0, v_0).

Computing the cross product, we obtain

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} & 0 \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} & 0 \end{vmatrix} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} \end{vmatrix} \mathbf{k} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} \mathbf{k}$$

This determinant is so important that it has its own terminology!

Definition The **Jacobian** of the transformation T given by $x = g(u, v)$ and $y = h(u, v)$ is

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

Therefore:

$$\vec{r_u} \times \vec{r_v} = \frac{\partial(x, y)}{\partial(u, v)}\vec{k}$$

So we can write the area as:

$$\Delta A \approx \left| \frac{\partial(x, y)}{\partial(u, v)}\vec{k} \right| \Delta u \Delta v$$

Since $\vec{k}$ is a unit vector, the magnitude of $\dfrac{\partial(x, y)}{\partial(u, v)}\vec{k}$ will be the absolute value of $\dfrac{\partial(x, y)}{\partial(u, v)}$:

$$\left| \frac{\partial(x, y)}{\partial(u, v)}\vec{k} \right| = \left| \frac{\partial(x, y)}{\partial(u, v)} \right|$$

Hence:

$$\Delta A \approx \left| \frac{\partial(x, y)}{\partial(u, v)} \right| \Delta u \Delta v$$

At the point $\left(u_0, v_0\right)$ this important formula relates the area of the region R and S. It tells us that *for small values of Δu and Δv, the area of R is approximately the absolute value of the Jacobian times the area of S.* In advanced calculus it's proven that this error approaches zero as Δu and Δv approach zero.

Now let's look at double integrals by dividing the regions into small rectangles and apply the Riemann sum:

$$\iint_R f(x,y)\,dA \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f\left(x_i, y_j\right)\Delta A$$

$$\approx \sum_{i=1}^{m}\sum_{j=1}^{n} f\left(x_i, y_j\right)\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\Delta u \Delta v$$
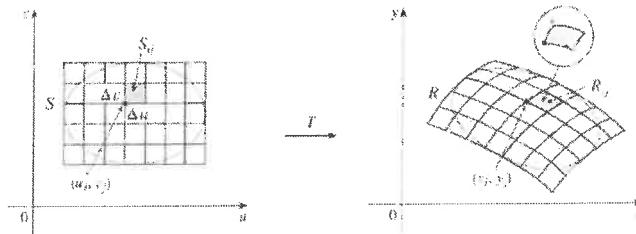
where x_i, y_j and the Jacobian are evaluated at $\left(u_i, v_j\right)$. This is the approximation for

$$\iint_S f(u,v)\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\Delta u \Delta v$$ so the following theorem can be used:

Change of Variables in a Double Integral Suppose that T is a C^1 transformation whose Jacobian is nonzero and that T maps a region S in the uv-plane onto a region R in the xy-plane. Suppose that f is continuous on R and that R and S are type I or type II plane regions. Suppose also that T is one-to-one, except perhaps on the boundary of S. Then

$$\iint_R f(x,y)\,dA = \iint_S f(x(u,v), y(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\,du\,dv$$

This theorem says that we change from an integral in of x and y to an integral in of u and v by expressing x and y in terms of u and v and writing $dA = \left|\dfrac{\partial(x,y)}{\partial(u,v)}\right|dudv$.

4. Examples continued

Ex2 <u>revisited</u>: Evaluate $I = \iint\limits_{D} \sin\left(9x^2 + 4y^2\right)dA$ where D is the region in the first quadrant bounded by

$9x^2 + 4y^2 = 1$

$$\text{Let } u = 3x \text{ and } v = 2y$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$$

$$= \begin{vmatrix} 3 & 0 \\ 0 & 2 \end{vmatrix}$$

$$= 3(2) - 0(0)$$

$$= 6$$

$$\text{we want } \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{6} \nwarrow \begin{array}{l}\text{scaling factor} \\ \text{is } abs\left(\frac{1}{6}\right)\end{array}$$

$$\downarrow$$

$$\Rightarrow I = \iint \sin(u^2 + v^2) \cdot \frac{1}{6} \, dA'$$

$$= \int_0^{\frac{\pi}{2}} \int_0^1 \sin(r^2) \cdot \frac{1}{6} \cdot r \, dr \, d\theta$$

$$= \frac{1}{6} \cdot \frac{\pi}{2} \left[-\frac{1}{2} \cos(r^2) \right]_0^1$$

$$= -\frac{\pi}{24} \left(\cos(1) - 1 \right)$$

$$= \frac{\pi}{24} \left(1 - \cos(1) \right)$$

Ex3 revisited: Set up an iterated integral to represent $I = \iint_D \dfrac{x-2y}{3x-y}\,dA$ where D is the parallelogram

enclosed by $x-2y=0$; $x-2y=4$; $3x-y=1$; and $3x-y=8$.

$$\text{Let } u = x - 2y \quad \text{and} \quad v = 3x - y$$

$$\Rightarrow \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$$

$$= \begin{vmatrix} 1 & -2 \\ 3 & -1 \end{vmatrix}$$

$$= 5$$

$$\Rightarrow \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{5} \quad (\text{scaling factor})$$
$$\text{is } Abs\left(\tfrac{1}{5}\right)$$

$$\Rightarrow I = \int_0^4 \int_1^8 \frac{u}{v} \cdot \frac{1}{5}\, dv\, du$$

$$= \frac{1}{5} \int_0^4 u\, du \cdot \int_1^8 \frac{dv}{v}$$

$$= \frac{1}{5} \left[\frac{u^2}{2} \right]_0^4 \left[\ln|v| \right]_1^8$$

$$= \frac{8}{5} \ln|8|$$

CONNECTION: rectangular $\longleftrightarrow$ polar coordinates

what is the scaling factor from (x, y) to (r, θ)

recall $x = r\cos\theta$ and $y = r\sin\theta$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta$$

$$= r \quad \longleftarrow \text{scaling factor}$$

Hence $dx\,dy = |r|\,dr\,d\theta$

we previously required $r \geq 0$

CONNECTION: rectangular ⟷ spherical coordinates

what is the scaling factor from (x,y,z) to (ρ, ϕ, θ)

recall $x = \rho \cos\theta \sin\phi ; \; y = \rho \sin\theta \sin\phi ; \; z = \rho \cos\phi$

$$\frac{\partial(x,y,z)}{\partial(\rho, \phi, \theta)} = \begin{vmatrix} x_\rho & x_\phi & x_\theta \\ y_\rho & y_\phi & y_\theta \\ z_\rho & z_\phi & z_\theta \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta\sin\phi & \rho\cos\theta\cos\phi & -\rho\sin\theta\sin\phi \\ \sin\theta\sin\phi & \rho\sin\theta\cos\phi & \rho\cos\theta\sin\phi \\ \cos\phi & -\rho\sin\phi & 0 \end{vmatrix}$$

$$= \cos\phi \begin{vmatrix} \rho\cos\theta\cos\phi & -\rho\sin\theta\sin\phi \\ \rho\sin\theta\cos\phi & \rho\cos\theta\sin\phi \end{vmatrix}$$

$$+ \rho\sin\phi \begin{vmatrix} \cos\theta\sin\phi & -\rho\sin\theta\sin\phi \\ \sin\theta\sin\phi & \rho\cos\theta\sin\phi \end{vmatrix}$$

$$= \cos\phi \left(\rho^2 \cos^2\theta \sin\phi\cos\phi + \rho^2 \sin^2\theta \sin\phi\cos\phi \right)$$

$$+ \rho\sin\phi \left(\rho\cos^2\theta\sin^2\phi + \rho\sin^2\theta\sin^2\phi \right)$$

$$= \rho^2 \sin\phi\cos^2\phi + \rho^2 \sin\phi\sin^2\phi$$

$$= \rho^2 \sin\phi$$

Hence $dx\,dy\,dz = \left| \rho^2 \sin\phi \right| d\rho\, d\phi\, d\theta$

we previously required $\rho \geq 0$ and $0 \leq \phi \leq \pi$.