HW. 51 min + 5Q's

Double Integrals in Polar Coordinates

Objective:

1. Double Integrals in Polar Coordinates

Important Formulas we will need:

RELATIONSHIP BETWEEN POLAR AND RECTANGULAR COORDINATES

1. To change from polar to rectangular coordinates, use the formulas

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta$$

2. To change from rectangular to polar coordinates, use the formulas

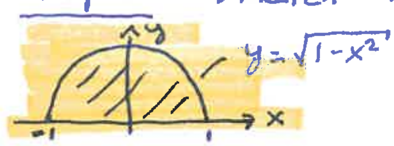
$$r^2 = x^2 + y^2 \quad \text{and} \quad \tan\theta = \frac{y}{x} \quad (x \neq 0)$$

1. *Double Integrals in Polar Coordinates*

In this section we will study double integrals in which the integrand and the region of integration are expressed in polar coordinates. Such integrals are important for two reasons: First, they arise naturally in many applications, and second, many double integrals in rectangular coordinates are more easily evaluated if they are converted to polar coordinates.

Step 1: sketch the region.

<u>Ex1</u>: Evaluate $I = \int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} \left(x^2+y^2\right)^{3/2}\, dy\,dx$



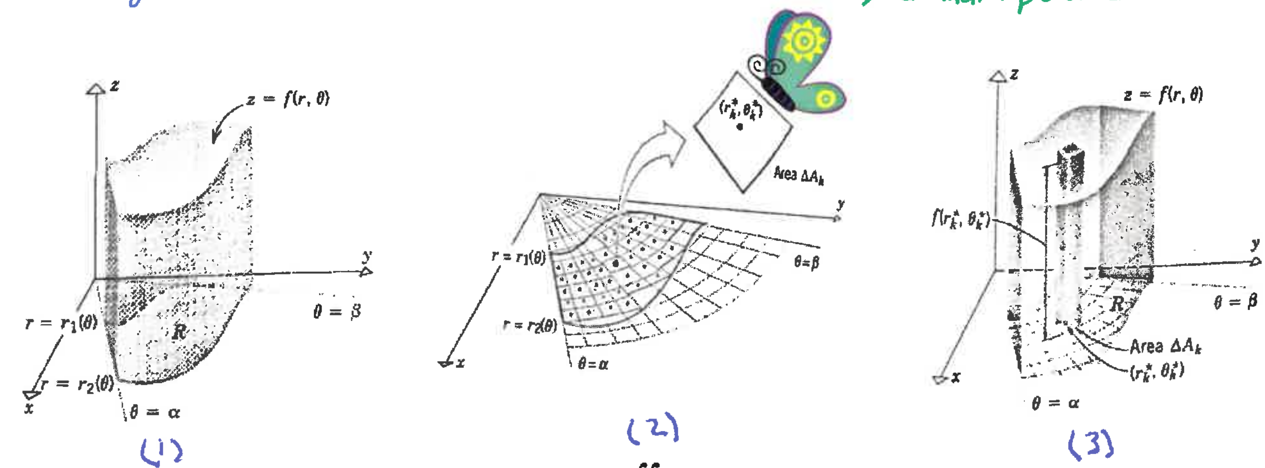
The region is a half disk. ... nice in polar coords.

Step 2: Antidifferentiate

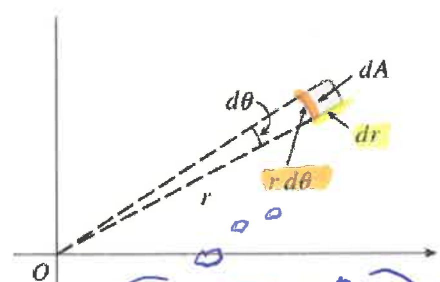
This would require trig sub @ the 1st step! who knows what would come next.
However, we notice that $x^2+y^2 = r^2$ ... nice in polar coords.

To find the volume of a solid bounded by the region $R=\{(r,\theta)|\ r_1(\theta)\le r\le r_2(\theta),\alpha\le\theta\le\beta\}$ in the xy-plane and a surface $z=f(r,\theta)$, we follow the same method as previous sections. (1) Dividing R into n small "polar" rectangles" (circular arcs and rays) of area ΔA_k, (2) picking a sample point in each rectangle, (3) Find the volume of each cylinder by $f(r_k^*,\theta_k^*)\Delta A_k$, (4) adding the volumes, (5) and finally allowing $n\to\infty$.

see manipulate 16.29

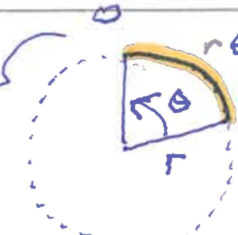


So the volume of the solid can be expressed as: $V=\iint\limits_R f(r,\theta)dA$. But how do we calculate dA (the area of the polar rectangles)?



we will calculate dA intuitively.

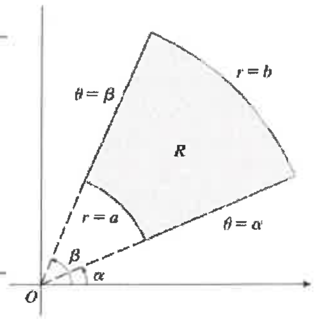
Consider the small polar rectangle pictured.



Notice when $\theta=2\pi$ we get circumference $=2\pi r$

Change to Polar Coodinates in a Double Integral If f is continuous on a polar rectangle R given by $0 \leqslant a \leqslant r \leqslant b$, $\alpha \leqslant \theta \leqslant \beta$, where $0 \leqslant \beta - \alpha \leqslant 2\pi$, then

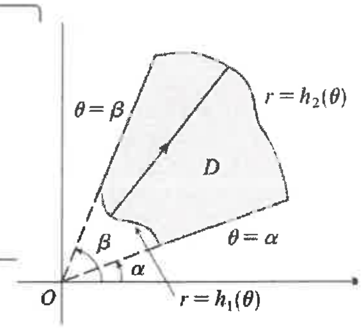
$$\iint_R f(x, y)\, dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$



If f is continuous on a polar region of the form

$$D = \{(r, \theta) \mid \alpha \leqslant \theta \leqslant \beta,\ h_1(\theta) \leqslant r \leqslant h_2(\theta)\}$$

then
$$\iint_D f(x, y)\, dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

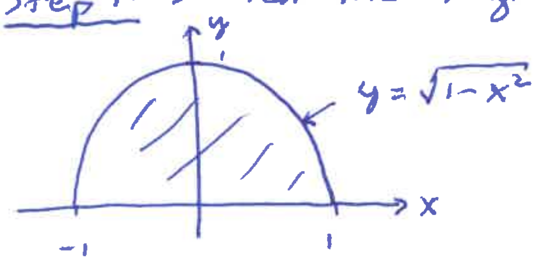


See manipulate 16, 34

Now let's go back to example 1 and use polar coordinates to evaluate it!

$$I = \int_{-1}^{1}\int_0^{\sqrt{1-x^2}} (x^2 + y^2)^{3/2}\, dy\,dx$$

Step 1: sketch the region



Step 2: find the limits of integration in polar coordinates

$0 \le r \le 1$

$0 \le \theta \le \pi$

Step 3: set up and evaluate the integral.

$$I = \int_0^\pi \int_0^1 \underbrace{(r^2)^{3/2}}_{r^3}\, r\, dr\, d\theta$$

$$= \int_0^\pi \underbrace{\int_0^1 r^4\, dr}_{\left[\frac{1}{5} r^5\right]_0^1}\, d\theta$$

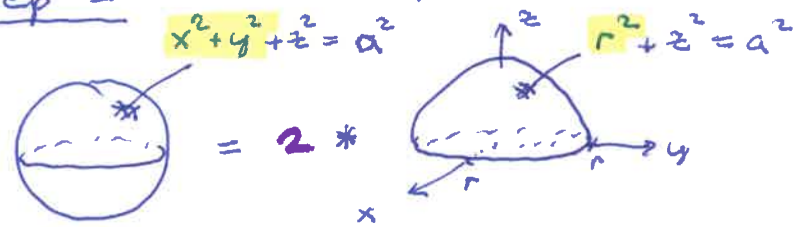
$$= \int_0^\pi \frac{1}{5}\, d\theta$$

$$= \frac{1}{5}\pi.$$

<u>Ex2</u>: Use a double polar integral to formulate the volume of a sphere of radius a.

$V = \frac{4}{3}\pi r^3$ volume of a sphere

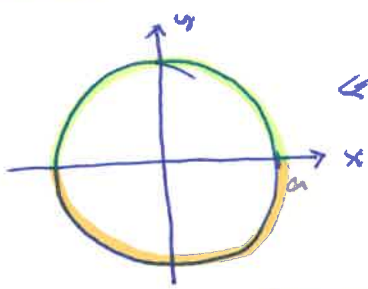
<u>Step 1</u>: Draw a picture



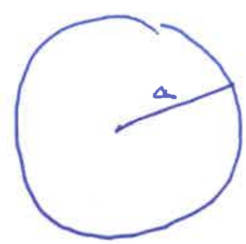
height
$z = \pm\sqrt{a^2-x^2-y^2}$

height
$z = +\sqrt{a^2-r^2}$

<u>Step 2</u>: Find the limits of integration



region over which we integrate



$-\sqrt{a^2-x^2} \le y \le \sqrt{a^2-x^2}$

$-a \le x \le a$

$0 \le r \le a$

$0 \le \theta \le 2\pi$

<u>Step 3</u>: Set up and evaluate the integral.

$$V = 2\int_0^{2\pi}\int_0^a \sqrt{a^2-r^2}\, r\, dr\, d\theta$$

recall
$dA = r\,dr\,d\theta$

no θ in the integrand

$$= 2\cdot 2\pi \int_0^a \sqrt{a^2-r^2}\, r\, dr\, d\theta$$

Let $u = a^2 - r^2$ $\quad u(0) = a^2$
$du = -2r\,dr$ $\quad u(a) = 0$

$$= 4\pi \int_{a^2}^{0} -\frac{1}{2}\sqrt{u}\, du$$

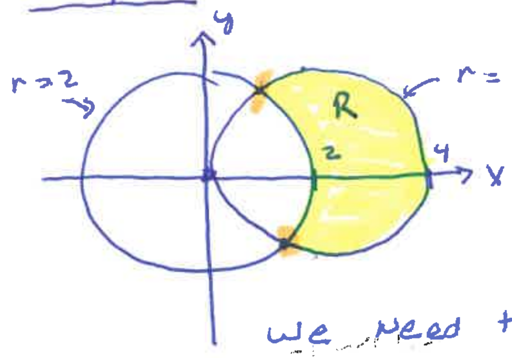
$$= -2\pi\left[\frac{2}{3}u^{3/2}\right]_{a^2}^{0}$$

And we see $V = \frac{4}{3}\pi a^3$.

setup an integral to represent

Ex3: $I = \iint_R g(r,\theta)\,dA$ where R is outside the circle $r=2$ and inside $r = 4\cos\theta$

Step 1: sketch a picture



we need the intersection points.

solve $2 = 4\cos\theta$

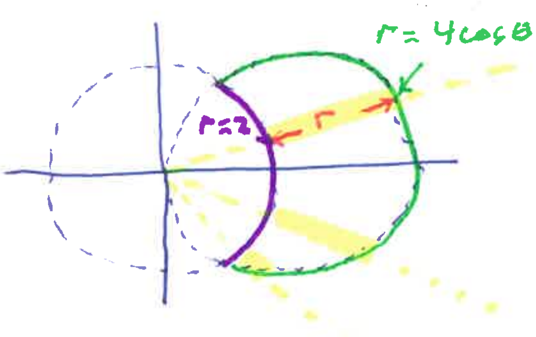
$\Rightarrow \frac{1}{2} = \cos\theta$

$\Rightarrow \theta = \frac{\pi}{3}$ or $\theta = -\frac{\pi}{3}$

or

$\theta = \frac{5}{3}\pi$

Step 2: Find the limits of integration.



$2 \le r \le 4\cos\theta$

$-\frac{\pi}{3} \le \theta \le \frac{\pi}{3}$

step 3: setup an integral

$$I = \iint_R g(r,\theta)\,dA$$

$$= \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \int_2^{4\cos\theta} g(r,\theta)\, r\,dr\,d\theta$$

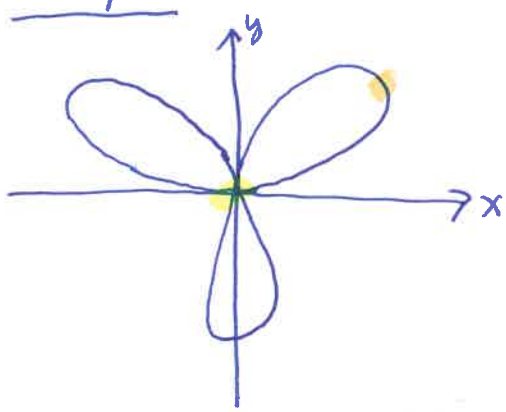
check out manipulate 16.35

As we saw in calc III, double integrals can be used to find area of a region:

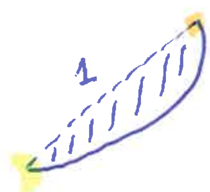
$$\text{area of } D = \iint_D 1\,dA$$

Ex4: Use a double polar integral to find the area enclosed by the three-petaled rose $r = \sin(3\theta)$

step 1: sketch the graph.



$=\ 6\ *$



step 2: Find the limits of integration.

solve: $\sin 3\theta = 0$

$\Rightarrow \theta = 0, \pm\frac{\pi}{3}, \pm\frac{2\pi}{3}, \pm\frac{3\pi}{3}, \ldots$

solve: $\sin 3\theta = 1$

$\Rightarrow 3\theta = \frac{\pi}{2} + 2k\pi$

$\Rightarrow \theta = \frac{\pi}{6} + \frac{2}{3}k\pi$

CAUTION: THERE IS A SUBTLE PITFALL HERE.

step 3: setup and evaluate an integral to represent the area.

$$A = 6\int_0^{\pi/6}\int_0^{\sin 3\theta} 1\, r\,dr\,d\theta$$

$$= 6\int_0^{\frac{\pi}{6}}\left[\frac{r^2}{2}\right]_0^{\sin 3\theta} d\theta$$

$$= 3\int_0^{\frac{\pi}{6}} \sin^2(3\theta)\,d\theta$$

$$\rightarrow A = 3\int_0^{\frac{\pi}{6}} \frac{1-\cos(6\theta)}{2}\,d\theta$$

$$= \frac{3}{2}\left[\theta - \frac{1}{6}\sin(6\theta)\right]_0^{\frac{\pi}{6}}$$

$$= \frac{3}{2}\cdot\frac{\pi}{6}$$

$$= \frac{\pi}{4}$$

This is the area of the polar rose.

Historical anecdote: There is a famous story about the nineteenth-century Scottish physicist Lord Kelvin. "Do you know what a mathematician is?" Kelvin once asked a class. He stepped to the blackboard and wrote:

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$$

"A mathematician," he continued, "is one to whom that is as obvious as 2*2 = 4 is to you." Let's explore just how obvious this really is …

Ex5: evaluate $I = \int_{-\infty}^{\infty} e^{-x^2}\,dx$

$$I = \int_{-\infty}^{\infty} e^{-x^2}\,dx$$

$$\Rightarrow I = 2\int_0^{\infty} e^{-x^2}\,dx$$

$$\Rightarrow I^2 = 4\left(\int_0^{\infty} e^{-x^2}\,dx\right)^2$$

$$= 4\int_0^{\infty} e^{-x^2}\,dx \int_0^{\infty} e^{-y^2}\,dy$$

$$= 4\int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)}\,dx\,dy$$

$$= 4\int_0^{\frac{\pi}{2}}\int_0^{\infty} e^{-r^2}\,r\,dr\,d\theta$$

$$= \frac{4}{-2}\cdot\frac{\pi}{2}\left[e^{-r^2}\right]_0^{\infty}$$

$$= -\pi(0-1)$$

$$= \pi$$

Thus $I^2 = \pi$

and $I = \int_{-\infty}^{\infty} e^{-x^2} = \sqrt{\pi}$

as Kelvin claimed.

Reflection on the anecdote: After working through this example it is clear that this formula is *not* obvious to your teacher or anyone who he knows. The conclusion seems to be that Kelvin was both showing off and trying to put down his class in a rather mean-spirited way.

Application: This integrand is called the normal or Gaussian distribution and is important in probability theory. The general form is $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$ where μ represents the mean and σ the standard deviation of a distribution. Some examples of (roughly) normally distributed variables include height, rolling a dice, tossing a coin, IQ, technical stock market, income distribution, shoe size, birth weight, and student grades .