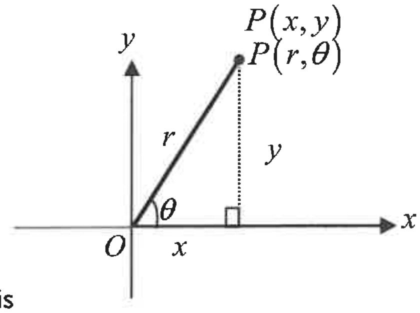
HW: 30 min + 5 Q's
54 min + 5 Q's

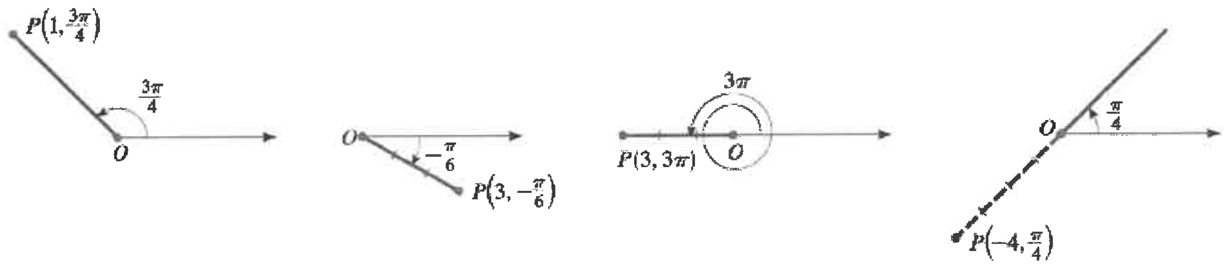
Polar Coordinates

We've been using the Cartesian (rectangular) coordinate system to graph. Another useful coordinate system is the **polar coordinate system** which uses distance and direction to specify the location of a point in the plane. The system is based on a point, called the **pole** (or origin) and a ray drawn in the direction of the x-axis, called the **polar axis**.



r is the distance from O to P.
θ is the angle between the polar axis and the segment $\overline{OP}$.

θ is positive if measures in a counterclockwise direction from the polar axis or negative if measured in a clockwise direction. If r is negative, then P is the point that is $|r|$ units from the pole in the direction opposite to that given by θ.

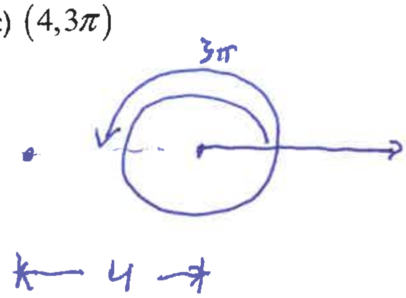


Ex1: Plot the given points.

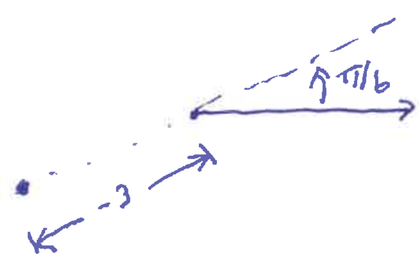
a) $\left(2, \frac{2\pi}{3}\right)$

b) $\left(1, -\frac{\pi}{4}\right)$

c) $(4, 3\pi)$

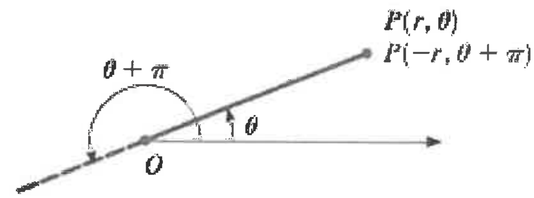


d) $\left(-3, \frac{\pi}{6}\right)$



Note: The coordinates (r,θ) and $(-r,\theta+\pi)$ represent the same point. Also keep in mind that because of the coterminal angles, each point has infinitely many representations. $P(r,\theta)$ is the same as:

$P(r,\theta+2n\pi)$ and $P(-r,\theta+(2n+1)\pi)$



Ex2: Graph $Q\left(1,\frac{5\pi}{4}\right)$ Find two other polar coordinate representations with $r>0$, and two with $r<0$.

$\left(1,\frac{5\pi}{4}\right.$

$\left(1,\frac{5\pi}{4}+2\pi\right)$ $\quad$ $\left(-1,\frac{5\pi}{4}-\pi\right)$

$\left(1,\frac{5\pi}{4}+4\pi\right)$ $\quad$ $\left(-1,\frac{5\pi}{4}+\pi\right)$

check out
manipulate
12, 20

RELATIONSHIP BETWEEN POLAR AND RECTANGULAR COORDINATES

1. To change from polar to rectangular coordinates, use the formulas

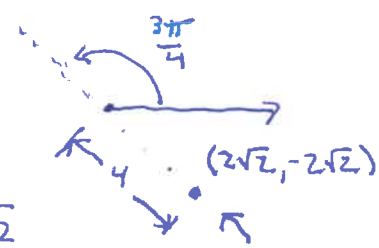
$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta$$

2. To change from rectangular to polar coordinates, use the formulas

$$r^2 = x^2 + y^2 \quad \text{and} \quad \tan\theta = \frac{y}{x} \quad (x \neq 0)$$

Note: These equations do not uniquely determine r or θ. Check to make sure you are in the correct quadrant.

Ex3: Find the rectangular coordinates of the point $\left(-4,\frac{3\pi}{4}\right)$.



in Quadrant IV

$$x = -4\cos\frac{3\pi}{4} = -4\left(-\frac{\sqrt{2}}{2}\right) = 2\sqrt{2}$$

$$y = -4\sin\frac{3\pi}{4} = -4\left(\frac{\sqrt{2}}{2}\right) = -2\sqrt{2}$$

$\boxed{\text{The quadrant is right correct from polar to rectangular.}}$

Ex4: Find the polar coordinates of the point $(-1, -\sqrt{3})$ ← Q3

$r = \sqrt{(-1)^2 + (\sqrt{3})^2} = 2$

$\theta = \tan^{-1}\left(\frac{-\sqrt{3}}{-1}\right) = \frac{\pi}{3}$

so at first glance the coordinates are $(2, \frac{\pi}{3})$ which is in quadrant 1. But we see that tangent > 0 in Q1 & Q3

so the point is $(2, \frac{4\pi}{3})$ ← $\frac{\pi}{3} + \pi$

$\boxed{\text{double check the quadrant when going from rect. to polar}}$

Ex5: Express the following equations in polar coordinates.

a) $x = 2$

$2 = r\cos\theta$

$\Rightarrow r = 2\sec\theta$

b) $x^2 + y^2 = 9$

$r^2 = 9$

$\Rightarrow r = 3$

Ex6: Express the following equations in rectangular coordinates.

a) $r = \dfrac{4}{1 + \sin\theta}$

$r = \dfrac{4}{1 + \sin\theta} = \dfrac{4}{1 + \frac{y}{r}} = \dfrac{4r}{r + y}$

$\Rightarrow \cancel{r}(r + y) = 4\cancel{r}$

$\Rightarrow r + y = 4$

$\Rightarrow (y - 4)^2 = (-r)^2$

$\Rightarrow \cancel{y^2} - 8y + 16 = r^2 = x^2 + \cancel{y^2}$

$\Rightarrow -8y + 16 = x^2$

$\Rightarrow -8y = x^2 - 16$

$\Rightarrow y = -\frac{1}{8}x^2 + 2$

b) $\tan\theta = 1$

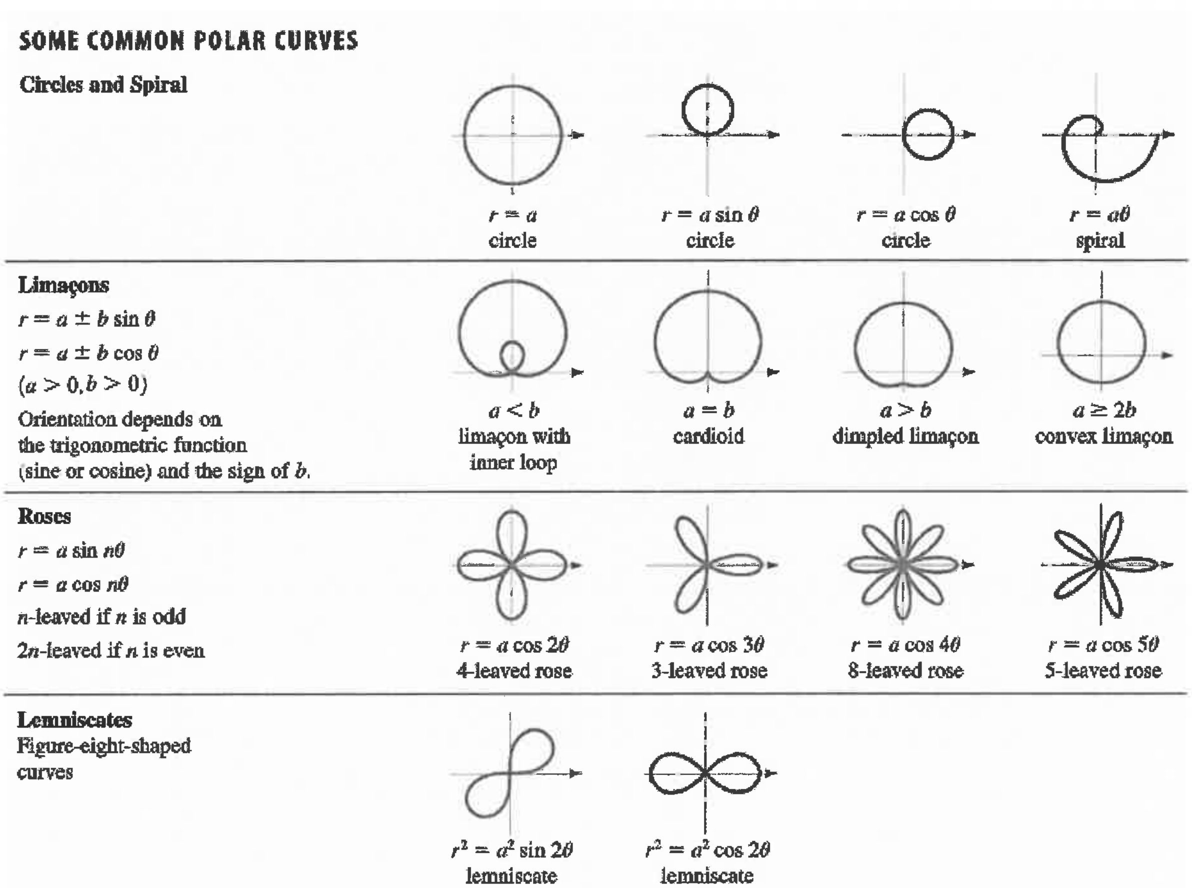
$\tan\theta = 1$

$\Rightarrow \dfrac{y}{x} = 1$

$\Rightarrow y = x$

❖ *Polar Curves*

The **graph of a polar equation** $r = f(\theta)$, or more generally $F(r, \theta) = 0$, consists of all points P that have at least one polar representation (r, θ) whose coordinates satisfy the equation.



❖ *Tangents to Polar Curves*

To find a tangent line to a polar curve $r = f(\theta)$, we regard θ as a parameter and write its parametric equations as

If given $(x(t), y(t))$ then $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}}$

$$x = r\cos\theta = f(\theta)\cos\theta \qquad y = r\sin\theta = f(\theta)\sin\theta$$

Then, using the method for finding slopes of parametric curves (from calc III) and the $\boxed{\text{Product Rule}}$, we have

u v u' v u v'

$$\boxed{\frac{dy}{dx} = \frac{\dfrac{dy}{d\theta}}{\dfrac{dx}{d\theta}} = \frac{\dfrac{d}{d\theta}(r\sin\theta)}{\dfrac{d}{d\theta}(r\cos\theta)} = \frac{\dfrac{dr}{d\theta}\sin\theta + r\cos\theta}{\dfrac{dr}{d\theta}\cos\theta - r\sin\theta}}$$

Note: $\dfrac{dy}{dx} \neq \dfrac{df}{d\theta}$

Notice that if we are looking for tangent lines at the pole, then $r = 0$ and this formula simplifies to:

$$\frac{dy}{dx} = \tan\theta \qquad \text{if} \quad \frac{dr}{d\theta} \neq 0$$

Ex9: Find the slope of the tangent line to $r = 1 - \cos\theta$ at θ.

$r = 1 - \cos\theta$

$\dfrac{dr}{d\theta} = \sin\theta$

so using the formula $\dfrac{dy}{dx} = \dfrac{r'\sin\theta + r\cos\theta}{r'\cos\theta - r\sin\theta}$

we have $\dfrac{dy}{dx} = \dfrac{\sin\theta \cdot \sin\theta + (1-\cos\theta)\cos\theta}{\sin\theta\cos\theta - (1-\cos\theta)\sin\theta}$

$$= \frac{\sin^2\theta + \cos\theta - \cos^2\theta}{\sin\theta\cos\theta - \sin\theta + \sin\theta\cos\theta}$$

$$= \frac{\cos\theta - \cos 2\theta}{\sin 2\theta - \sin\theta}$$

check out manipulate 12.35

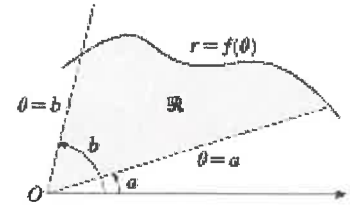
when $\theta = 2\pi$ this becomes $A = \frac{1}{2} r^2 \cdot 2\pi$ or $A = \pi r^2$ which is the area of a circle.

check out manipulate 12,36

❖ ***Area in Polar Coordinates***

Recall the area of a sector: $A = \frac{1}{2} r^2 \theta$

Now suppose that we want to find the area of the shaded region on the given graph where f is positive and continuous and $0 < b - a \le 2\pi$.



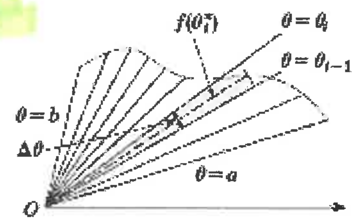
Divide $[a, b]$ into subintervals of equal width $\Delta\theta$. Then:

$$\Delta A_i \approx \frac{1}{2}\left[f\left(\theta_i^*\right)\right]^2 \Delta\theta$$

← $\Delta A = \frac{1}{2} r^2 \Delta\theta$ with r replaced by $f(\theta_i^*)$

$$A \approx \sum_{i=1}^{n} \frac{1}{2}\left[f\left(\theta_i^*\right)\right]^2 \Delta\theta$$

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{2}\left[f\left(\theta_i^*\right)\right]^2 \Delta\theta = \int_a^b \frac{1}{2}\left[f(\theta)\right]^2 d\theta$$

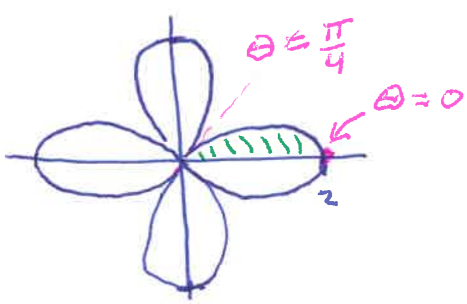


This is often written as:

$$\boxed{A = \int_a^b \frac{1}{2} r^2 \, d\theta}$$

Ex10: Find the area of the region in the plane enclosed by the polar rose $r = 2\cos 2\theta$

Step 1: sketch the region.



Step 2: Find the limits of integration.

solve $0 = 2\cos 2\theta$

$\Rightarrow 0 = \cos 2\theta$

$\Rightarrow \theta = \frac{\pi}{4} + k\frac{\pi}{2}$

solve $2 = 2\cos 2\theta$

$\Rightarrow \theta = 0 + 2k\pi$

step 3: set-up the integral and integrate.

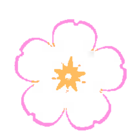
$$A = 8 \cdot \frac{1}{2} \int_0^{\pi/4} (2\cos 2\theta)^2 \, d\theta$$

$$= 16 \int_0^{\pi/4} \cos^2 2\theta \, d\theta$$

$$= 16 \int_0^{\pi/4} \frac{1 + \cos 4\theta}{2} \, d\theta$$

$$= 8 \left[\theta + \frac{1}{4}\sin 4\theta\right]_0^{\pi/4}$$

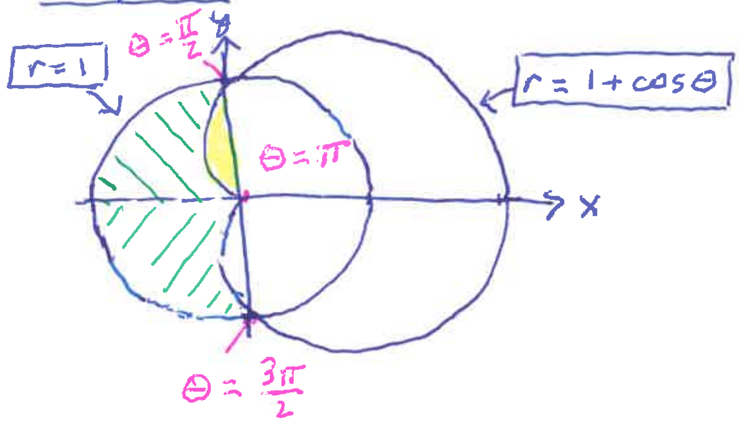
$$= 2\pi$$



check out manipulate 12.38

Ex11: Find the area of the region that lies inside the circle $r=1$ and outside the cardioid $r=1+\cos\theta$.

Step 1: sketch the region



Step 2: find the limits of integration

solve $1 = 1+\cos\theta$

$\Rightarrow 0 = \cos\theta$

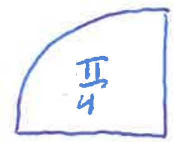
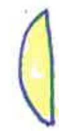
$\Rightarrow \theta = \frac{\pi}{2} + k\pi$

solve $0 = 1+\cos\theta$

$\Rightarrow -1 = \cos\theta$

$\Rightarrow \theta = \pi + 2k\pi$

Step 3: set-up the integral and integrate.

$\frac{1}{2}A =$  $-$ 

$$= \frac{\pi}{4} - \int_{\frac{\pi}{2}}^{\pi} \frac{1}{2}(1+\cos\theta)^2\, d\theta$$

$$= \frac{\pi}{4} - \frac{1}{2}\int_{\frac{\pi}{2}}^{\pi} 1 + 2\cos\theta + \cos^2\theta\, d\theta$$

$$= \frac{\pi}{4} - \frac{1}{2}\int_{\frac{\pi}{2}}^{\pi} 1 + 2\cos\theta + \frac{1+\cos 2\theta}{2}\, d\theta$$

$$= \frac{\pi}{4} - \frac{1}{2}\left[\frac{3}{2}\theta + 2\sin\theta + \frac{\sin 2\theta}{4}\right]_{\frac{\pi}{2}}^{\pi}$$

$$= \frac{\pi}{4} - \left(\frac{3}{8}\pi - 1\right)$$

$$\Rightarrow \frac{1}{2}A = 1 - \frac{\pi}{8}$$

AND $A = 2 - \frac{\pi}{4}$

check out manipulate 12.40

❖ *Length of the Polar Curves*

To find the length of a polar curve $r = f(\theta)$, $a \leq \theta \leq b$, we regard θ as a parameter and write the $\boxed{\text{parametric equations}}$ of the curve as

$$x = r\cos\theta = f(\theta)\cos\theta \qquad y = r\sin\theta = f(\theta)\sin\theta$$

Using the Product Rule and differentiating with respect to θ, we obtain

$$\frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta \qquad \frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta$$

If given $(x(t), y(t))$ then the arclength is found with

$$L = \int_{t=a}^{t=b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

(from calc III)

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = \left(\frac{dr}{d\theta}\right)^2 \cos^2\theta - 2r\frac{dr}{d\theta}\cos\theta\,\sin\theta + r^2\sin^2\theta$$

$$+\left(\frac{dr}{d\theta}\right)^2 \sin^2\theta + 2r\frac{dr}{d\theta}\sin\theta\,\cos\theta + r^2\cos^2\theta$$

$$= \left(\frac{dr}{d\theta}\right)^2 + r^2$$

Assuming that f' is continuous, we can use our theorem from calc II to write:

$$L = \int_a^b \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

Therefore the length of a curve with polar equation $r = f(\theta)$, $a \leqslant \theta \leqslant b$, is

$$\boxed{L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta}$$

Ex12: Find the length of the cardioid $r = 1 + \cos\theta$.

$r = 1 + \cos\theta$ and $\frac{dr}{d\theta} = -\sin\theta$

$$\Rightarrow L = \underset{\text{symmetry}}{2} \int_0^{\pi} \sqrt{(1+\cos\theta)^2 + (-\sin\theta)^2}\, d\theta$$

$$= 2\int_0^{\pi} \sqrt{1 + 2\cos\theta + \underbrace{\cos^2\theta + \sin^2\theta}_{1}}\, d\theta$$

$$= 2\int_0^{\pi} \sqrt{\underbrace{2 + 2\cos\theta}_{4\left(\frac{1+\cos\theta}{2}\right)}}\, d\theta$$

$$= 2 \cdot 2 \int_0^{\pi} \sqrt{\cos^2\left(\frac{\theta}{2}\right)}\, d\theta$$

$$= 4\int_0^{\pi} \cos\left(\frac{\theta}{2}\right) d\theta$$

$\Rightarrow L = 8\left[\sin\frac{\theta}{2}\right]_0^{\pi}$

$= 8$ (arclength)