❖ *Alternating Series*

The convergence tests that we have looked at so far apply only to series with positive terms. In this section and the next we learn how to deal with series whose terms are not necessarily positive. Of particular importance are *alternating series,* whose terms alternate in sign.

A series in which the terms are alternately positive and negative is an **alternating series**. Here are two examples:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1}\frac{1}{n}$$

$$-\frac{1}{2} + \frac{2}{3} - \frac{3}{4} + \frac{4}{5} - \frac{5}{6} + \frac{6}{7} - \cdots = \sum_{n=1}^{\infty} (-1)^{n}\frac{n}{n+1}$$

Pro-tip: Try a few terms to determine if you have $+-+-\ldots$ OR $-+-+\ldots$

As you can see, the series could start with a positive or a negative term. So the nth term of an alternating series could be described as:

$$a_n = (-1)^{n-1} b_n \text{ or } a_n = (-1)^{n} b_n \text{ where } b_n = |a_n|.$$

Let's explore alternating series using a famous example.

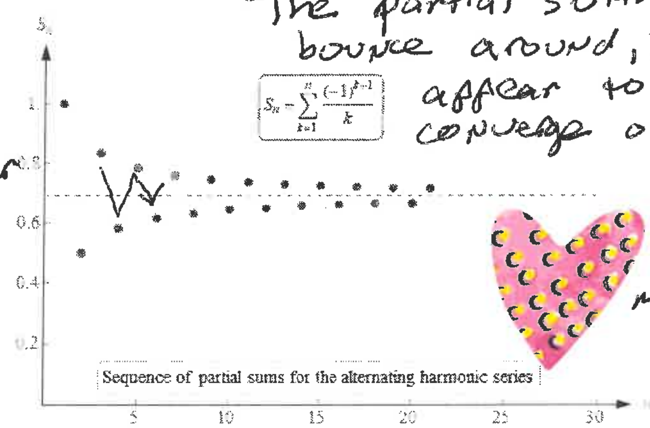
Ex1: The alternating harmonic series.

$$\sum_{n=1}^{\infty} (-1)^{n-1}\frac{1}{n} \text{ (the alternating harmonic series)}$$

We investigate this question by looking at the sequence of partial sums for the series. In this case, the first four terms of the sequence of partial sums and graph are as follows:

$$S_1 = 1$$
$$S_2 = 1 - \frac{1}{2} = \frac{1}{2}$$
$$S_3 = 1 - \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$$
$$S_4 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} = \frac{7}{12}$$

smaller
bigger
smaller



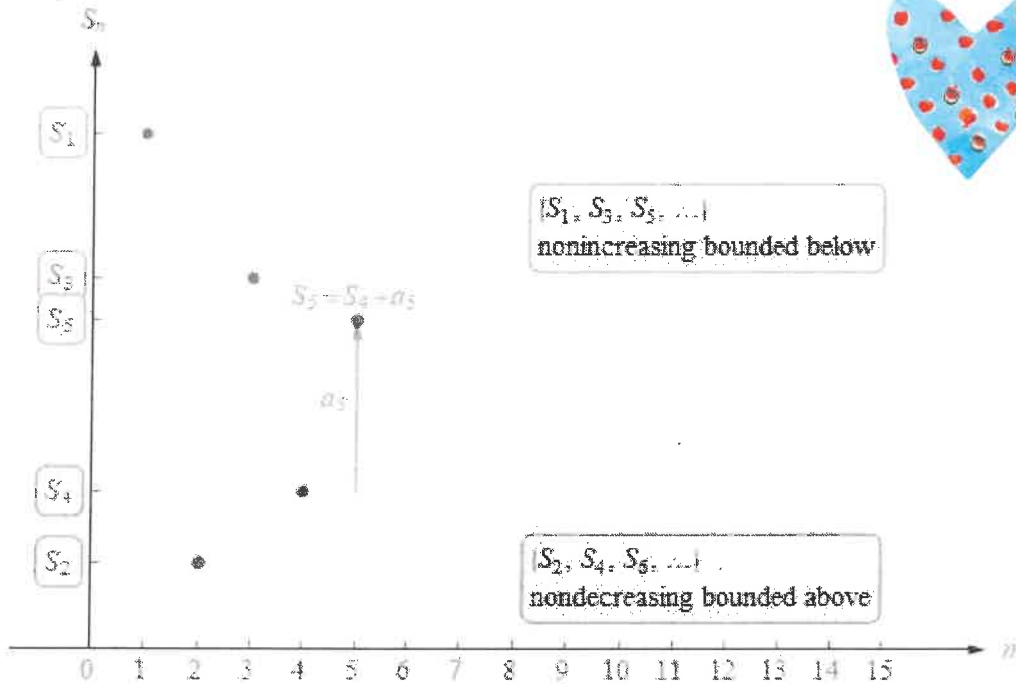
The partial sums bounce around, but appear to converge on 0.69.

check out manipulate 10.34

THE LIST: (1.) The geometric series converges when $|r| < 1$. (2.) The harmonic series diverges. (3.) Telescoping series. (4.) The integral test. (5.) The test for divergence. (6.) The p-series converges for $p > 1$. (7.) The comparison test (weak). (8.) The limit comparison test (stronger). (9.) The alternating harmonic series converges.

just a hunch at this point.

While we only have an intuitive sense that the alternating harmonic series converges, we are beginning to see a pattern.



You will
love ST
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The picture can be formalized with the alternating series test.

Alternating Series Test If the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \cdots \qquad b_n > 0$$

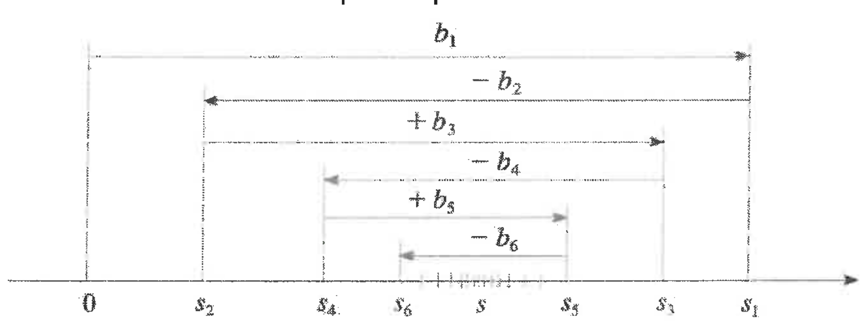
satisfies

(i) $b_{n+1} \leqslant b_n$ for all n

(ii) $\lim_{n \to \infty} b_n = 0$

then the series is convergent.

A second picture proof is as follows:



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❖ ***Absolute Convergence***

Given any series Σa_n, we can consider the corresponding series

$$\sum_{n=1}^{\infty} |a_n| = |a_1| + |a_2| + |a_3| + \cdots$$

whose terms are the absolute values of the terms of the original series.

> **1 Definition** A series Σa_n is called **absolutely convergent** if the series of absolute values $\Sigma |a_n|$ is convergent.

Notice that if Σa_n is a series with positive terms, then $|a_n| = a_n$ and so absolute convergence is the same as convergence in this case.

Ex2: Are the following series convergent or divergent? If convergent, are they absolutely convergent?

a) $\displaystyle\sum_{n=0}^{\infty}(-1)^n \frac{1}{2^n}$

$\displaystyle\sum_{n=0}^{\infty} \left|(-1)^n \frac{1}{2^n}\right| = \sum_{n=0}^{\infty} \frac{1}{2^n}$ which is a convergent geometric series,

$\therefore \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{1}{2^n}$ converges absolutely.

b) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1} \frac{1}{\sqrt{n}}$

$\displaystyle\sum_{n=1}^{\infty} \left|(-1)^{n-1} \frac{1}{\sqrt{n}}\right| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is a divergent p-series,

$\therefore$ the series is not absolutely convergent,

$\displaystyle\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$ so $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{\sqrt{n}}$ is convergent by the alternating series test.

> $\boxed{2}$ **Definition** A series $\Sigma\, a_n$ is called **conditionally convergent** if it is convergent but not absolutely convergent.

Ex3: Is the series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{\sqrt{n}}$ convergent or divergent? If convergent, is it conditionally or absolutely convergent?

As shown in ex 2b, the series is convergent, but not absolutely convergent. Thus it is conditionally convergent.

From these examples you can see that it is possible for a series to be convergent but not absolutely convergent. Convergence does not provide absolute convergence. On the other hand …

> $\boxed{3}$ **Theorem** If a series $\Sigma\, a_n$ is absolutely convergent, then it is convergent.

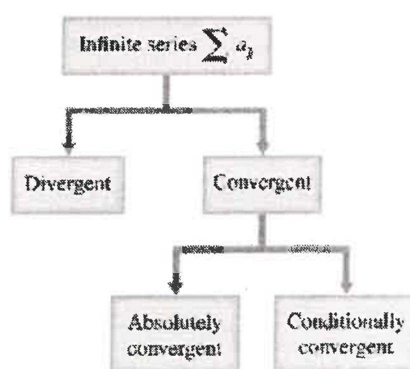
Ex4: Is the following series convergent?

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^3}$$

$$\sum_{n=1}^{\infty}\left|(-1)^{n+1}\frac{1}{n^3}\right| = \sum_{n=1}^{\infty}\frac{1}{n^3}$$ is a convergent p-series.

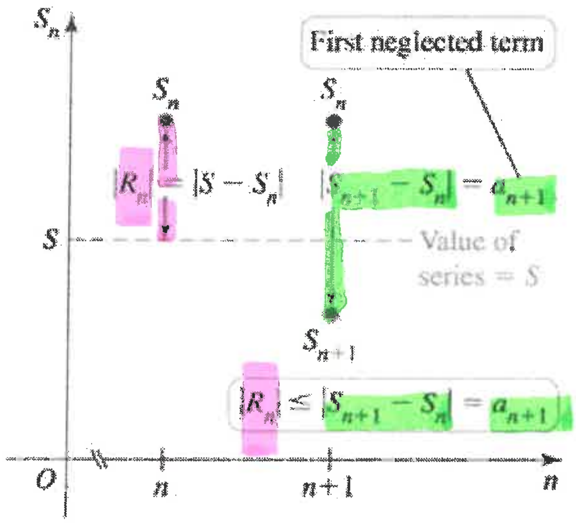
thus $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^3}$ is absolutely convergent and convergent.

Sometimes it can be helpful to represent the various options using a flowchart.



Recall that if a series converges to a value S, then the remainder is $R_n = S - S_n$, where S_n is the sum of the first n terms of the series. The magnitude of the remainder is the *absolute error* in approximating S by S_n.

An upper bound on the magnitude of the remainder in an alternating series arises from the following observation: When the terms are non-increasing in magnitude, the value of the series is always trapped between successive terms of the sequence of partial sums



$|R_N| \le a_{N+1}$

Theorem: Remainder in Alternating Series

$\{a_k \geq 0$

Let $\sum_{k=1}^{\infty}(-1)^{k+1} a_k$ be a convergent alternating series with terms that are non-increasing in magnitude. Let $R_n = S - S_n$ be the remainder in approximating the value of that series by the sum of its first n terms. Then $|R_n| \le a_{n+1}$. In other words, the magnitude of the remainder is less than or equal to the magnitude of the first neglected term.

Ex5: Using power series, you can show that $\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k} = \ln(2)$. How many terms of the series are required to approximate $\ln(2)$ with error less than 10^{-6}?

Solve: $|R_N| \le \frac{1}{N+1} < 10^{-6}$ The next term

$\Rightarrow \frac{1}{1000001} > \frac{1}{N}$

$\Rightarrow N > 1000001$

over one million terms are required (which is not very efficient).

Making a connection: When trying to understand exponential growth, one tool is to look at the growth rate. For example, during the COVID pandemic, we tracked the number of positive cases in Washington State.

Date	Cases	Weekly Growth	Ratio $\frac{a_{n+1}}{a_n}$
1-Mar	30	245 − 30	
8-Mar	245	215	
15-Mar	921	676	3.14
22-Mar	2234	1313	1.94
29-Mar	5112	2878	2.19
5-Apr	8145	3033	3033/2878 = 1.05
12-Apr	10360	2215	0.73
19-Apr	12107	1747	0.79
26-Apr	13724	1617	0.93

Data from the WA dept of health,

Ratios that are over 1 indicate continued growth and divergence (in the case of COVID that was very bad). Ratios under 1 indicate that growth is slowing and convergence (good). A ratio of exactly 1 is inconclusive leaving the possibility of either convergence or divergence.

❖ ***The Ratio and Root Test***

The Ratio Test

(i) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).

(ii) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1$ or $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, the Ratio Test is inconclusive; that is, no conclusion can be drawn about the convergence or divergence of $\sum a_n$.

THE LIST: (1.) The geometric series converges when $|r|<1$. (2.) The harmonic series diverges. (3.) Telescoping series. (4.) The integral test. (5.) The test for divergence. (6.) The p-series converges for $p>1$. (7.) The comparison test (weak). (8.) The limit comparison test (stronger). (9.) The alternating harmonic series converges. (10.) The alternating series test. (11.) The ratio test.

Ex6: Do the following series converge or diverge? If they converge, is it conditional or absolute convergence?

a) $\sum_{n=1}^{\infty}(-1)^n \frac{2^n}{n!}$ use the ratio test

$$\lim_{N\to\infty}\left|\frac{(-1)^{N+1}2^{N+1}}{(N+1)!}\cdot\frac{N!}{(-1)^N 2^N}\right|$$

$$= \lim_{N\to\infty}\frac{2}{N+1}$$

$$= 0 < 1 \quad \text{(less than 1)}$$

$\therefore$ The series is absolutely convergent by the ratio test.

b) $\sum_{n=1}^{\infty}\frac{N^N}{N!}$ use the ratio test

$$\lim_{N\to\infty}\frac{(N+1)^{N+1}}{(N+1)!}\cdot\frac{N!}{N^N}$$

$$= \lim_{N\to\infty}\frac{(N+1)^{N+1}}{(N+1)N^N}$$

$$= \lim_{N\to\infty}\left(\frac{N+1}{N}\right)^N$$

$$\Rightarrow = \lim_{N\to\infty}\left(1+\frac{1}{N}\right)^N \to 1^\infty$$

$$= \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$$

$$= e^{\lim_{N\to\infty} x\ln(1+\frac{1}{x})}$$

$$= e^{\lim_{N\to\infty}\frac{\ln(1+\frac{1}{x})}{1/x}}$$

$$\overset{H}{=} e^{\lim_{N\to\infty}\frac{\frac{1}{1+\frac{1}{x}}\cdot\left(-\frac{1}{x^2}\right)}{\left(-\frac{1}{x^2}\right)}}$$

$$= e^1 > 1$$

Therefore the series diverges by the ratio test.

Ex7: Show that both convergence and divergence are possible when $L = 1$ by considering the p-series

(A) $\sum_{n=1}^{\infty} n^2$ and (B) $\sum_{n=1}^{\infty} n^{-2} = \sum_{N=1}^{\infty}\frac{1}{N^2}$

(A) divergent p-series

(B) convergent p series.

(A) Ratio test

$$\lim_{N\to\infty}\frac{(N+1)^2}{N^2}$$

$$= \lim_{N\to\infty}\frac{N^2+2N+1}{N^2}$$

$$= 1$$

so the ratio test is inconclusive

(B) Ratio test

$$\lim_{N\to\infty}\frac{N^2}{(N+1)^2} = 1$$

the ratio test is inconclusive.

The ratio test fails w/ p-series.

The Root Test

(i) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).

(ii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$ or $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$, the Root Test is inconclusive.

Ex 8: Do the following series converge or diverge? If convergent, is it conditional or absolute?

a) $\sum_{n=1}^{\infty}\left(\frac{n}{2n+3}\right)^n$ use the root test.

$$\lim_{n\to\infty} \sqrt[n]{\left|\left(\frac{n}{2n+3}\right)^n\right|} = \lim_{n\to\infty} \frac{n}{2n+3} = \frac{1}{2} < 1$$

The series converges absolutely by the root test.

b) $\sum_{n=3}^{\infty}\left(-\frac{(n+1)^2}{3n-6}\right)^n$ use the root test

$$\lim_{n\to\infty} \sqrt[n]{\left|\left(-\frac{(n+1)^2}{3n-6}\right)^n\right|} = \lim_{n\to\infty} \frac{(n+1)^2}{3n-6} = \infty > 1$$

The series diverges by the root test.

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You may have wondered what the big deal is with conditionally convergent series. Let's explore a mind blowing example using a rearrangement.

Recall: $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots = \ln(2)$

multiply both sides by $\frac{1}{2}$.

$$\Rightarrow \quad \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \dots = \frac{1}{2}\ln(2)$$

insert 0 between the terms

$$0 + \frac{1}{2} + 0 - \frac{1}{4} + 0 + \frac{1}{6} + 0 - \frac{1}{8} + \dots = \frac{1}{2}\ln(2)$$

AND $\quad 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \dots = \ln(2)$

sum to: $\quad 1 + 0 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + 0 + \frac{1}{7} - \frac{1}{4} + \dots = \frac{3}{2}\ln(2)$

OR: $\quad 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \dots = \frac{3}{2}\ln(2)$

The same infinite series has two sums if we simply rearrange the terms!

More generally, Riemann proved:

If $\sum a_n$ is a conditionally convergent series and r is <u>any</u> real number, then there is a rearrangement of $\sum a_n$ that sums to r!