The Comparison Tests

The Comparison Test Suppose that Σa_n and Σb_n are series with positive terms.
(i) If Σb_n is convergent and $a_n \leqslant b_n$ for all n, then Σa_n is also convergent.
(ii) If Σb_n is divergent and $a_n \geqslant b_n$ for all n, then Σa_n is also divergent.

In using the Comparison Test we must, of course, have some known series Σb_n for the purpose of comparison. Most of the time we use one of these series:

- A p-series [$\Sigma 1/n^p$ converges if $p > 1$ and diverges if $p \leqslant 1$;
- A geometric series [Σar^{n-1} converges if $|r| < 1$ and diverges if $|r| \geqslant 1$;

Ex1: Determine the convergence of the following.

a) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}3^n} \leq \sum_{n=1}^{\infty} \frac{1}{3^n}$ which is a convergent geometric series.

$\therefore \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}\,3^n}$ converges by the comparison test.

b) $\sum_{n=1}^{\infty} \frac{1}{(n^2+3)^{1/3}} \leq \sum_{n=1}^{\infty} \frac{1}{n^{2/3}}$ Divergent p-series.

The test is inconclusive.

visualize this with:

$\boxed{\text{known series}}$

	BIGGER	SMALLER
CONVERGES	unknown converges	?
DIVERGES	?	unknown diverges

THE LIST: (7) The comparison test (weak)

NOTE 1 Although the condition $a_n \leqslant b_n$ or $a_n \geqslant b_n$ in the Comparison Test is given for all n, we need verify only that it holds for $n \geqslant N$, where N is some fixed integer, because the convergence of a series is not affected by a finite number of terms.

Ex2: Determine the convergence of the following.

$$\sum_{n=1}^{\infty} \frac{n^2}{n^4+n-10} = \sum_{n=1}^{10} \frac{n^2}{n^4+n-10} + \sum_{n=11}^{\infty} \frac{n^2}{n^4+n-10}$$

$$< \quad " \quad + \sum_{n=11}^{\infty} \frac{1}{n^2} \quad \text{a convergent p-series.}$$

$\therefore$ The series converges by the comparison test.

NOTE 2 The terms of the series being tested must be smaller than those of a convergent series or larger than those of a divergent series. If the terms are larger than the terms of a convergent series or smaller than those of a divergent series, then the Comparison Test doesn't apply.

To be able to answer the last problem we use the following variation of the Direct Comparison test.

The Limit Comparison Test Suppose that $\Sigma\, a_n$ and $\Sigma\, b_n$ are series with positive terms. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

where c is a finite number and $c > 0$, then either both series converge or both diverge.

A couple special cases:

- If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.
- If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

Back to Ex2: $\sum_{n=1}^{\infty} \frac{n^2}{n^4+n-10}$

consider $\lim_{n\to\infty} \frac{\frac{1}{n^2}}{\frac{n^2}{n^4+n-10}} = \lim_{n\to\infty} \frac{n^4 - n - 1}{n^4} = 1$

since the limit is a number the two series share the same fate.

$\therefore$ since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (a p-series) we know $\sum_{n=1}^{\infty} \frac{n^2}{n^4+n-10}$ converges by the limit comparison test.

Back to Ex1b: $\sum_{n=1}^{\infty} \frac{1}{(n^2+3)^{1/3}}$

consider $\lim_{n\to\infty} \frac{\frac{1}{(n^2+3)^{1/3}}}{\frac{1}{n^{2/3}}} = \lim_{n\to\infty} \frac{n^{2/3}}{(n^2+3)^{1/3}} = 1$

$\therefore$ since $\sum_{n=1}^{\infty} \frac{1}{n^{2/3}}$ diverges (a p-series) we know $\sum_{n=1}^{\infty} \frac{1}{(n^2+3)^{1/3}}$ diverges by the limit comparison test.