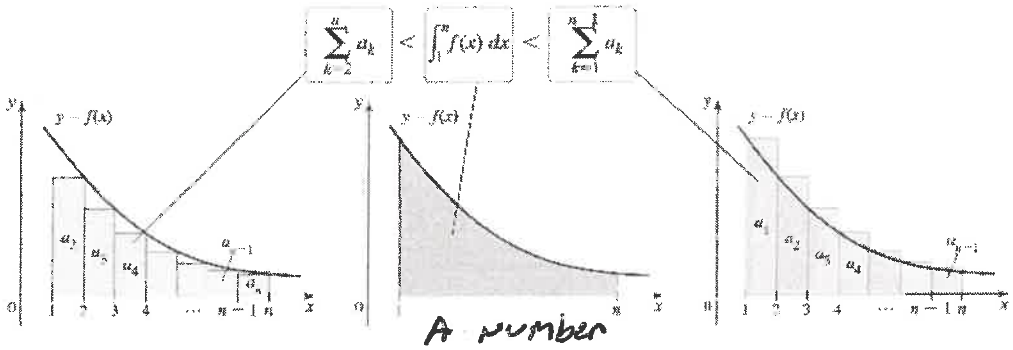
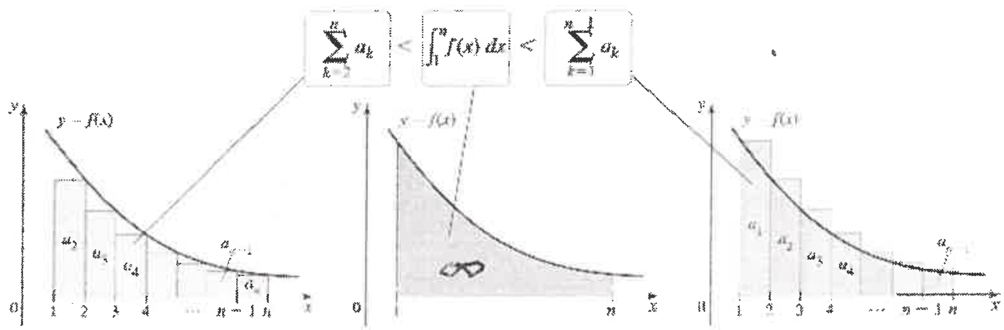
The Integral Test

Case 1: When the integral (gray area) converges, so does the left sum.



Case 2: When the integral (gray area) diverges, so does the right sum.



The Integral Test Suppose f is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)\,dx$ is convergent. In other words:

(i) If $\int_1^{\infty} f(x)\,dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.

(ii) If $\int_1^{\infty} f(x)\,dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

NOTE We should *not* infer from the Integral Test that the sum of the series is equal to the value of the integral. In fact,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \qquad \text{whereas} \qquad \int_1^{\infty} \frac{1}{x^2}\,dx = 1$$

Therefore, in general,

$$\sum_{n=1}^{\infty} a_n \neq \int_1^{\infty} f(x)\,dx$$

NOTE When we use the Integral Test, it is not necessary to start the series or the integral at $n = 1$. For instance, in testing the series

$$\sum_{n=4}^{\infty} \frac{1}{(n-3)^2} \qquad \text{we use} \qquad \int_4^{\infty} \frac{1}{(x-3)^2}\,dx$$

Also, it is not necessary that f be *always* decreasing. What is important is that f be *ultimately* decreasing, that is, decreasing for x larger than some number N. Then $\sum_{n=N}^{\infty} a_n$ is convergent, so $\sum_{n=1}^{\infty} a_n$ is convergent.

<u>Ex1</u>: Using the integral test, show (again) that the "Harmonic Series" $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

$$\int_1^{\infty} \frac{dx}{x} = \lim_{t\to\infty} \int_1^t \frac{dx}{x}$$

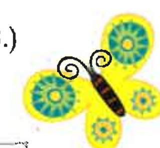
$$= \lim_{t\to\infty} \Big[\ln|x|\Big]_1^t$$

$$= \lim_{t\to\infty} \left(\overset{\infty}{\ln|t|} - \ln|1|\right)$$

$$= \infty$$

Since $\int_1^{\infty} \frac{dx}{x}$ diverges, $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by the integral test.

THE LIST: (1.) The geometric series converges when $|r| < 1$. (2.) The harmonic series diverges. (3.) Telescoping series. (4.) The integral test.



Theorem If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$.

check out manipulate 10.25

NOTE 1 With any *series* $\sum a_n$ we associate two *sequences*: the sequence $\{s_n\}$ of its partial sums and the sequence $\{a_n\}$ of its terms. If $\sum a_n$ is convergent, then the limit of the sequence $\{s_n\}$ is s (the sum of the series) and the limit of the sequence $\{a_n\}$ is 0.

NOTE 2 The converse is not true in general. If $\lim_{n\to\infty} a_n = 0$, we can not conclude that $\sum a_n$ is convergent. Observe that for the harmonic series $\sum 1/n$ we have $a_n = 1/n \to 0$ as $n \to \infty$, but we showed that $\sum 1/n$ is divergent.

7 **Test for Divergence** If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

NOTE 3 If we find that $\lim_{n\to\infty} a_n \neq 0$, we know that $\sum a_n$ is divergent. If we find that $\lim_{n\to\infty} a_n = 0$, we know *nothing* about the convergence or divergence of $\sum a_n$. Remember the warning in Note 2: If $\lim_{n\to\infty} a_n = 0$, the series $\sum a_n$ might converge or it might diverge.

Ex2: Show that the series $\sum_{n=1}^{\infty} \frac{-n}{2n+5}$ diverges.

$$\lim_{n\to\infty} \frac{-n}{2n+5} = -\frac{1}{2} \neq 0$$

$\therefore \sum_{n=1}^{\infty} \frac{-n}{2n+5}$ diverges by the test for divergence.

THE LIST: (1.) The geometric series converges when $|r|<1$. (2.) The harmonic series diverges. (3.) Telescoping series. (4.) The integral test. (5.) The test for divergence.

One simple but powerful application of the integral test is what we call the *p-series*. This will be a powerful tool for us in the next section when we learn about comparison tests.

❖ **p-series**

p-series are series of the form $\sum_{n=1}^{\infty} \frac{1}{n^p}$ In which p is a fixed number. $\sum_{n=1}^{\infty} \frac{1}{n^p} \begin{cases} converges & if \quad p>1 \\ diverges & if \quad p\le 1 \end{cases}$

1 The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$.

Ex3: Do the following series converge or diverge?

a) $\sum_{n=1}^{\infty} \frac{6}{n^5} = 6\sum_{n=1}^{\infty} \frac{1}{n^5}$ This is a convergent p-series $(p=5)$

b) Repeat the same example using the integral test: $\sum_{n=1}^{\infty}\frac{6}{n^5}$

we could also do this w/ the integral test

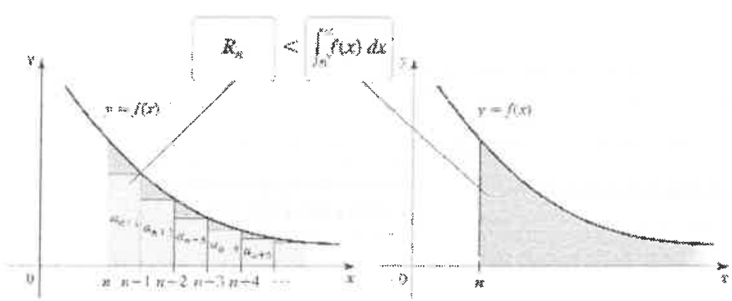
$\int_1^{\infty}\frac{6}{x^5}\,dx = \lim_{t\to\infty}\left[-\frac{6}{4x^4}\right]_1^t \quad \frac{6}{4}$

since the integral converges, the series converges by the integral test.

c) $\sum_{n=1}^{\infty}\frac{10}{\sqrt[5]{n}} = 10\sum_{n=1}^{\infty}\frac{1}{n^{1/5}}$ This is a divergent p-series $(p=\frac{1}{5})$

THE LIST: (1.) The geometric series converges when $|r|<1$. (2.) The harmonic series diverges. (3.) Telescoping series. (4.) The integral test. (5.) The test for divergence. (6.) The p-series converges for $p>1$.

The last item in this section is our first error bounding technique. It is computationally convenient to approximate infinite series with partial sums. But when approximating, it is important to know the worst case scenario (maximum error). One way to do this is to bound the remainder using an improper integral.



For example, we can use this technique to show that we need $n>1000$ terms to guarantee that $\sum_{k=1}^{\infty}\frac{1}{k^2}=\sum_{k=1}^{n}\frac{1}{k^2}\pm 0.001$.