HW: 33 min + 2Q
25 min + 2Q

Sequences

A **sequence** can be thought of as a list of numbers written in a definite order:

$$a_1,\ a_2,\ a_3,\ a_4,\ \ldots,\ a_n,\ \ldots$$

The number a_1 is called the *first term,* a_2 is the *second term,* and in general a_n is the *nth term.* We will deal exclusively with infinite sequences and so each term a_n will have a successor a_{n+1}.

Notice that for every positive integer n there is a corresponding number a_n and so a sequence can be defined as a function whose domain is the set of positive integers. But we usually write a_n instead of the function notation $f(n)$ for the value of the function at the number n.

NOTATION The sequence $\{a_1, a_2, a_3, \ldots\}$ is also denoted by

$$\{a_n\} \qquad \text{or} \qquad \{a_n\}_{n=1}^{\infty}$$

When a_n is given by a formula we refer to it as the general term of a sequence.

- ***Arithmetic Sequences*** (linear)

A sequence is arithmetic if there is a common difference, d, between two consecutive terms so $a_n = a_{n-1} + d$.

__Ex1__: 2, 5, 8, ...

a) What is d? $d = 3$ (increases by 3)

b) What is a_1? $a_1 = 2$

c) What is the 4th term of this sequence? $a_4 = 11$

➢ The nth term of an arithmetic sequence is given by: $a_n = a_1 + (n-1)d$

d) What is the 20th term of this sequence?

$$a_{20} = 2 + (20-1)(3) = 59$$

- ***Geometric Sequences*** (exponential)

A sequence is geometric if there is a common ratio, r, between two consecutive terms so $\dfrac{a_{n+1}}{a_n} = r$

__Ex2__: 3, 6, 12, ...

a) What is r? $r = 2$

b) What is a_1? $a_1 = 3$

c) What is the 4th term of this sequence? $a_4 = 24$

➢ The nth term of a geometric sequence is given by: $a_n = a_1 r^{n-1}$

d) What is the 20th term of this sequence?

$$a_{20} = 3 \cdot 2^{19} = 1572864$$

Ex3: Determine whether each sequence is arithmetic or geometric. Find the next term.

a) 1, 3, 5, ... Arithmetic $a_4 = 7$ $(d = 2)$

b) 2, 4, 8, ... Geometric $a_4 = 16$ $(r = 2)$

c) 6, 3, 1.5, ... Geometric $a_4 = 0.75$ $(r = \frac{1}{2})$

d) 12, 7, 2, −3, ... Arithmetic $a_4 = -8$ $(d = -5)$

e) 3, −30, 300, −3000, ... Geometric $a_4 = +30{,}000$ $(r = -10)$

Ex4: What is the general term of the sequence $\left\{\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1, \frac{5}{4}, \frac{3}{2}, \ldots\right\}$? How else can you present this sequence?

Arithmetic $a_1 = \frac{1}{4}$ and $d = \frac{1}{4}$

$a_n = \frac{1}{4} + \frac{1}{4}(n-1)$ OR $a_n = \frac{1}{4} n$

linear w/ slope $= \frac{1}{4}$ & "y-int" $= 0$

Ex5: Write out the first few terms of the sequence $\left\{(-1)^n \sqrt{n}\right\}_{n=3}^{\infty}$

$-\sqrt{3}, 2, -\sqrt{5}, +\sqrt{6}, \ldots$

Not all sequences are generated by a formula. For instance the sequence $\{3,1,4,1,5,9,2,6,\ldots\}$ is the digits of π, and there is no formula for the nth digit of π.

A very famous sequence is called ~~a~~ the *Fibonacci sequence*. Can you come up with its pattern? $\{1,1,2,3,5,8,13,21,\ldots\}$

recursive formula

$f_1 = 1$

$f_2 = 1$

$f_n = f_{n-1} + f_{n-2}$ for $n = 3, 4, 5, \ldots$

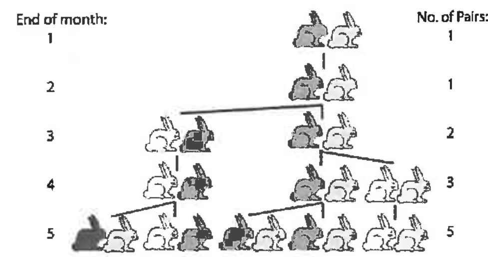
explicit formula

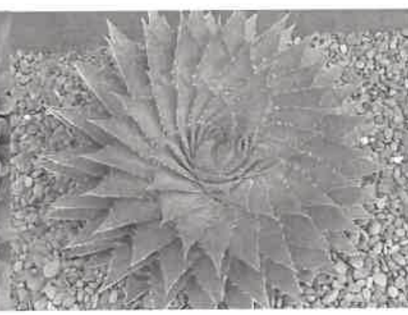
$$f_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$$

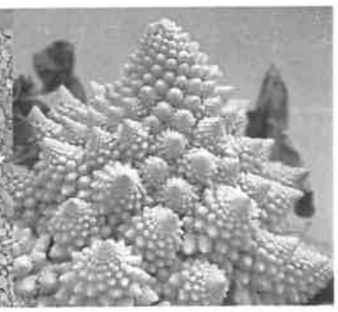
You can derive this using linear algebra and diagonalization.

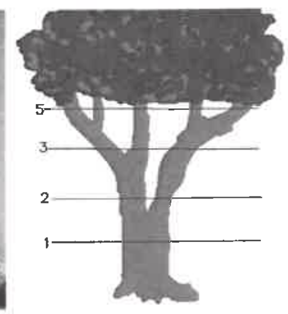
This sequence is defined recursively. This means the first one or two terms may be given and other terms are found by a pattern applied to the preceding terms.

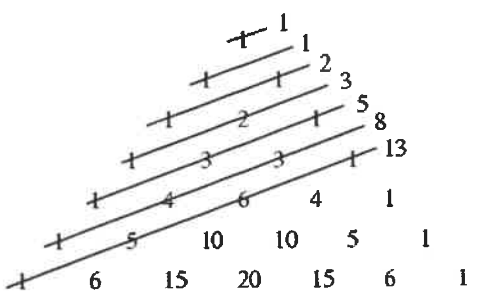
This sequence arose when the 13th-centry Italian mathematician ~~known as~~ Fibonacci ~~solved~~ posed a problem concerning the breeding of rabbits. This particular sequence appears in a surprisingly wide variety of situations, particularly in nature. For instance, the number of spiral arms in a sunflower almost always turns out to be a number from this sequence.

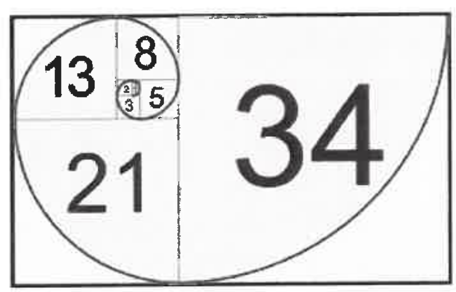


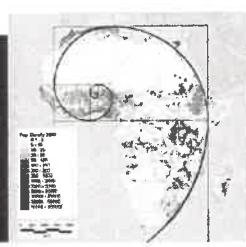


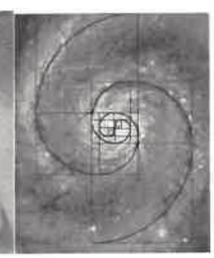












Now consider the sequence $a_n = \dfrac{n}{n+1}$ and picture it two ways:

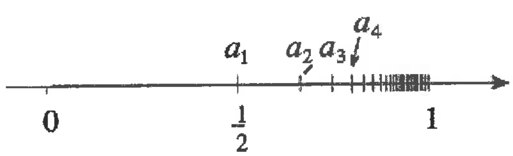


Figure 1

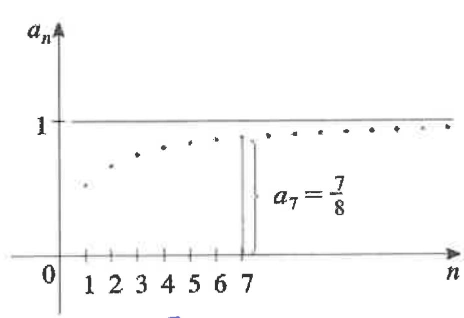


Figure 2

Note that, since a sequence is a function whose domain is the set of positive integers, its graph consists of isolated points with coordinates

$$(1, a_1) \qquad (2, a_2) \qquad (3, a_3) \qquad \ldots \qquad (n, a_n) \qquad \ldots$$

From Figure 1 or Figure 2 it appears that the terms of the sequence $a_n = n/(n+1)$ are approaching 1 as n becomes large. In fact, the difference

$$1 - \frac{n}{n+1} = \frac{1}{n+1}$$

can be made as small as we like by taking n sufficiently large. We indicate this by writing

$$\lim_{n \to \infty} \frac{n}{n+1} = 1$$

In general, the notation

$$\lim_{n \to \infty} a_n = L$$

means that the terms of the sequence $\{a_n\}$ approach L as n becomes large. Notice that the following definition of the limit of a sequence is very similar to the definition of a limit of a function at infinity given in Section 2.6.

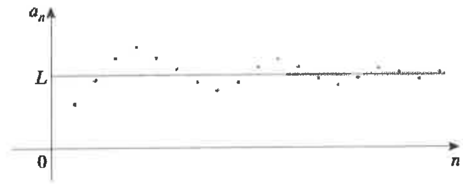
1 Definition A sequence $\{a_n\}$ has the **limit** L and we write

$$\lim_{n \to \infty} a_n = L \qquad \text{or} \qquad a_n \to L \text{ as } n \to \infty$$

if we can make the terms a_n as close to L as we like by taking n sufficiently large. If $\lim_{n \to \infty} a_n$ exists, we say the sequence **converges** (or is **convergent**). Otherwise, we say the sequence **diverges** (or is **divergent**).

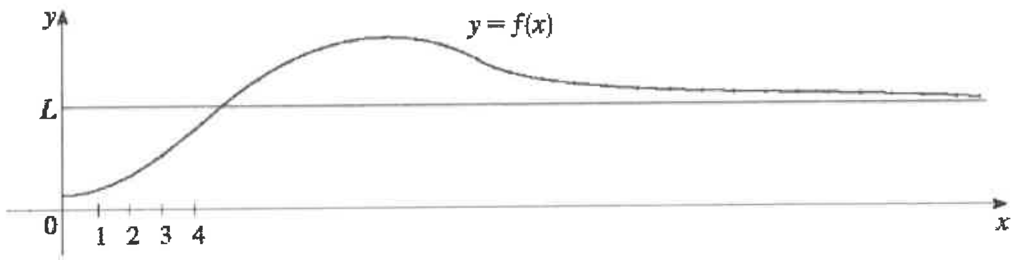
Ex6: Find $\lim_{n \to \infty} (n+1) = \infty$ sequence (diverges)

The following are examples of two sequences that converge to L:



simple, but powerful thm

3 **Theorem** If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$ when n is an integer, then $\lim_{n\to\infty} a_n = L$.



In particular, since we know that $\lim_{x\to\infty} (1/x^r) = 0$ when $r > 0$ (~~Theorem 2.6.5~~ calc I,), we have

n replaces x

4 $$\lim_{n\to\infty} \frac{1}{n^r} = 0 \qquad \text{if } r > 0$$

If a_n becomes large as n becomes large, we use the notation $\lim_{n\to\infty} a_n = \infty$.

If $\lim_{n\to\infty} a_n = \infty$, then the sequence $\{a_n\}$ is divergent but in a special way. We say that $\{a_n\}$ diverges to ∞.

The Limit Laws given in ~~Section 2.3~~ calc I also hold for the limits of sequences and their proofs are similar.

If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is a constant, then

$$\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} (a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} ca_n = c \lim_{n\to\infty} a_n \qquad\qquad \lim_{n\to\infty} c = c$$

$$\lim_{n\to\infty} (a_n b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} \quad \text{if } \lim_{n\to\infty} b_n \neq 0$$

$$\lim_{n\to\infty} a_n^p = \left[\lim_{n\to\infty} a_n\right]^p \text{ if } p > 0 \text{ and } a_n > 0$$

Ex7: Find the following limits.

a) $\lim\limits_{n\to\infty}(-1)^n$ Diverges. The sequence is: $-1, 1, -1, 1, -1, \ldots$

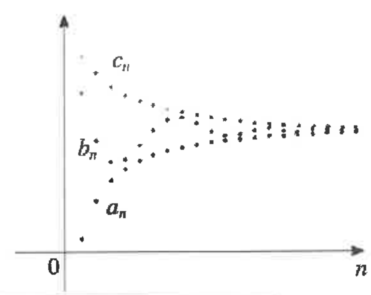
b) $\lim\limits_{n\to\infty}\frac{n+4}{n+1} = \lim\limits_{n\to\infty}\frac{1+\frac{4}{n}}{1+\frac{1}{n}} = 1$

c) $\lim\limits_{n\to\infty}\frac{n+\ln n}{n^2} = \lim\limits_{x\to\infty}\frac{x+\ln x}{x^2} \overset{H}{=} \lim\limits_{x\to\infty}\frac{1+\frac{1}{x}}{2x} = 0$

l'Hospital's Rule requires derivatives and thus continuous $f(x)$

The Squeeze Theorem can also be adapted for sequences as follows

If $a_n \leqslant b_n \leqslant c_n$ for $n \geqslant n_0$ and $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} c_n = L$, then $\lim\limits_{n\to\infty} b_n = L$.



6 Theorem If $\lim\limits_{n\to\infty}|a_n| = 0$, then $\lim\limits_{n\to\infty} a_n = 0$.

Proof:

$-|a_n| \leq a_n \leq |a_n|$ for all n.

And $\lim\limits_{n\to\infty} -|a_n| = \lim\limits_{n\to\infty} +|a_n| = 0$

Thus $\lim\limits_{n\to\infty} a_n = 0$ by the squeeze thm.

Q.E.D.

check out manipulate 10.19

> $\boxed{7}$ **Theorem** If $\lim_{n\to\infty} a_n = L$ and the function f is continuous at L, then
>
> $$\lim_{n\to\infty} f(a_n) = f(L)$$

Ex8: Find the following limits.

a) $\lim_{n\to\infty} e^{\frac{3n}{n+1}} = e^{\lim_{n\to\infty} \frac{3n}{n+1}}$

$= e^{3}$

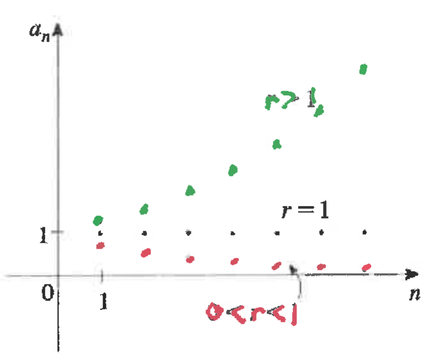
b) $\lim_{n\to\infty} \sqrt{\frac{n+1}{n}} = \sqrt{\lim_{n\to\infty} \frac{n+1}{n}}$

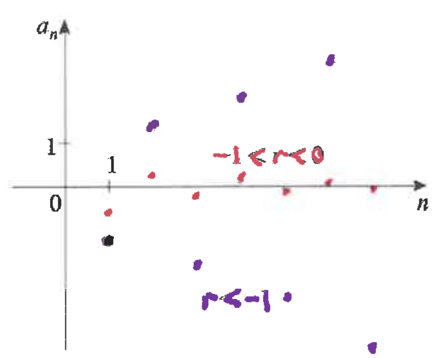
$= 1$

$\left(\frac{1}{2}\right)^n$ and $\left(-\frac{1}{3}\right)^n$

Note: $\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \\ \infty & \text{if } r > 1 \\ DNE & \text{if } r \le -1 \end{cases}$

2^n





10 **Definition** A sequence $\{a_n\}$ is called **increasing** if $a_n < a_{n+1}$ for all $n \geqslant 1$, that is, $a_1 < a_2 < a_3 < \cdots$. It is called **decreasing** if $a_n > a_{n+1}$ for all $n \geqslant 1$. A sequence is **monotonic** if it is either increasing or decreasing.

11 **Definition** A sequence $\{a_n\}$ is **bounded above** if there is a number M such that

$$a_n \leqslant M \qquad \text{for all } n \geqslant 1$$

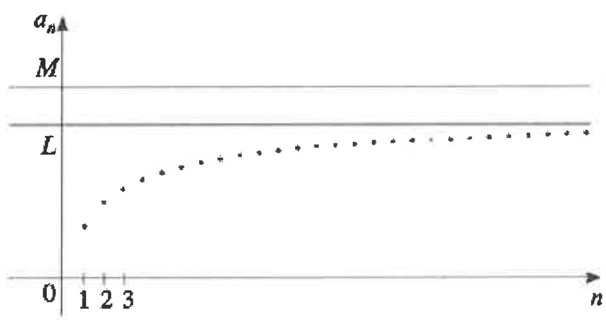
It is **bounded below** if there is a number m such that

$$m \leqslant a_n \qquad \text{for all } n \geqslant 1$$

If it is bounded above and below, then $\{a_n\}$ is a **bounded sequence**.

For instance, the sequence $a_n = n$ is bounded below $(a_n > 0)$ but not above. The sequence $a_n = n/(n+1)$ is bounded because $0 < a_n < 1$ for all n.

We know that not every bounded sequence is convergent [for instance, the sequence $a_n = (-1)^n$ satisfies $-1 \leqslant a_n \leqslant 1$ but is divergent from Example 7a and not every monotonic sequence is convergent $(a_n = n \to \infty)$. But if a sequence is both bounded *and* monotonic, then it must be convergent. Intuitively you can understand why it is true by looking at the pic If $\{a_n\}$ is increasing and $a_n \leqslant M$ for all n, then the terms are forced to crowd together and approach some number L.



check out manipulate 10.21

12 **Monotonic Sequence Theorem** Every bounded, monotonic sequence is convergent.

Ex9: Verify that $a_n = \sqrt{n+1} - \sqrt{n}$ is decreasing and bounded below. Does $\lim_{n\to\infty} a_n$ exist?

We can see $\sqrt{n+1} - \sqrt{n} > 0$ so the sequence is bounded below.

consider $f(x) = \sqrt{x+1} - \sqrt{x}$

$$\Rightarrow f'(x) = \frac{1}{2\sqrt{x+1}} - \frac{1}{2\sqrt{x}}$$

$$= \frac{\sqrt{x} - \sqrt{x+1}}{2\sqrt{x+1}\sqrt{x}}$$

$$< 0$$

so $f(x)$ is decreasing for $x > 0$ and thus $a_n = f(n)$ is decreasing.

Since the sequence is bounded below and decreasing, it converges by the monotonic sequence theorem.

Ex10: Show that the following sequence is bounded and increasing. Then prove that $L = \lim_{n\to\infty} a_n$ exists and compute its value. calculate $\lim_{N\to\infty} a_N$ if

$$a_1 = \sqrt{2},\quad a_2 = \sqrt{2\sqrt{2}},\quad a_3 = \sqrt{2\sqrt{2\sqrt{2}}},\ldots$$

$\frac{1}{2} = 1 - \frac{1}{2}$

$$a_1 = \sqrt{2} = 2^{1/2}$$

$\frac{1}{2} + \frac{1}{4} = 1 - \frac{1}{4}$

$$a_2 = \sqrt{2\sqrt{2}} = 2^{\frac{1}{2}} \cdot 2^{\frac{1}{4}} = 2^{\frac{1}{2}+\frac{1}{4}}$$

equals $1 - \frac{1}{8}$

$$a_3 = \sqrt{2\sqrt{2\sqrt{2}}} = 2^{\frac{1}{2}} \cdot 2^{\frac{1}{4}} \cdot 2^{\frac{1}{8}} = 2^{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}}$$

check out manipulate 10.9

$$\vdots$$

$$a_N = 2^{1 - \frac{1}{2^N}}$$

And $$\lim_{N\to\infty} 2^{1-\frac{1}{2^N}} = 2^{\lim_{N\to\infty}\left(1 - \frac{1}{2^N}\right)} = 2^1 = 2$$