Stokes' Theorem

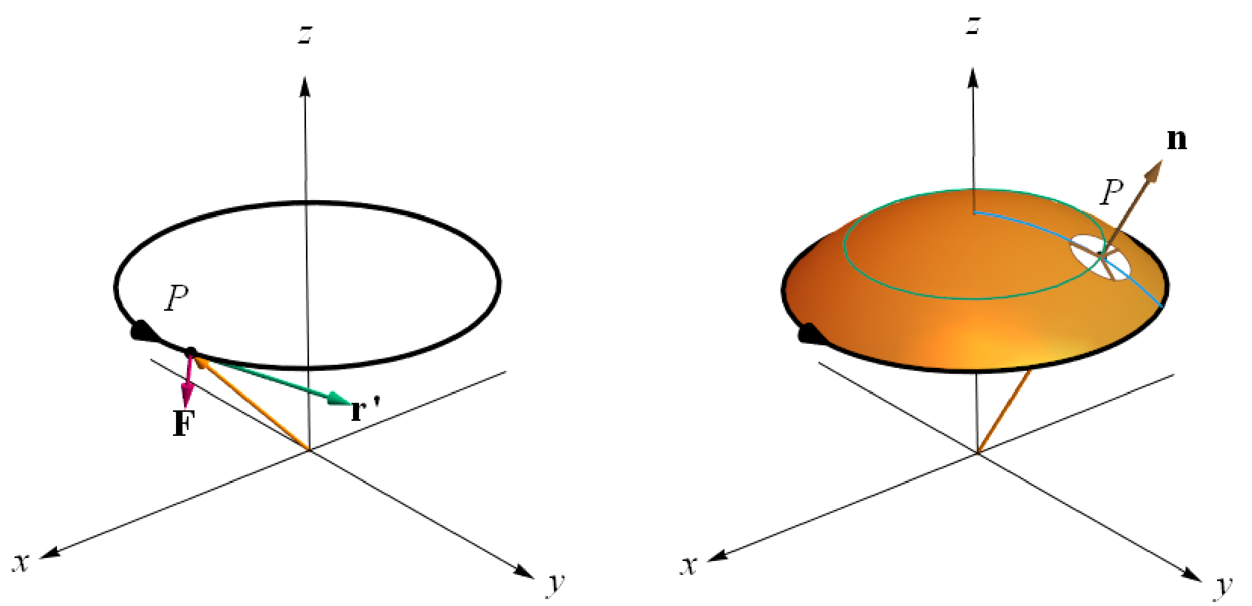
With the divergence, the curl, and surface integrals in hand, we are ready to present two of the crowning results of calculus. Fortunately, all of the heavy lifting has been done. In this section, you will see Stokes' Theorem.

Stokes' Theorem is the three-dimensional version of the circulation form of Green's Theorem.

$$\text{circulation} = \oint_C \vec{F}\cdot d\vec{r} = \iint_R \underbrace{\left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)}_{\text{2D curl}} dA \text{ (Green's Theorem)}$$

In Stokes' Theorem, the plane region *R* in Green's Theorem becomes an oriented surface *S* in $\mathbb{R}^3$. The circulation integral in Green's Theorem remains a circulation integral, but now over the closed simple piecewise-smooth oriented curve *C* that forms the boundary of *S*. The double integral of the curl in Green's Theorem becomes a surface integral of the three-dimensional curl.

$$\text{circulation} = \oint_C \vec{F}\cdot d\vec{r} = \iint_S \underbrace{\left(\nabla\times\vec{F}\right)}_{\text{3D curl}}\cdot \vec{n}\, dS \text{ (Stokes' Theorem)}$$



Stokes' Theorem involves an oriented curve C and an oriented surface S on which there are two unit normal vectors at every point. These orientations must be consistent and the normal vectors must be chosen correctly. Here is the right-hand rule that relates the orientations of S and C, and determines the choice of the normal vectors.

A common situation occurs when C has a counterclockwise orientation when viewed from above; then, the vectors normal to S point upward.

Theorem: Stokes' Theorem

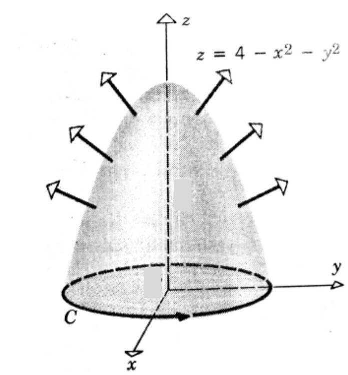
Let *S* be an oriented surface in $\mathbb{R}^3$ with a piecewise-smooth closed boundary *C* whose orientation is consistent with that of *S*. Assume $\vec{F} = \langle f, g, h \rangle$ is a vector field whose components have continuous first partial derivatives on *S*. Then

$$\text{circulation} = \oint_C \vec{F} \cdot d\vec{r} = \iint_S \left(\nabla \times \vec{F}\right) \cdot \vec{n}\, dS$$

Where $\vec{n}$ is the unit vector normal to S determined by the orientation of S.

The meaning of Stokes' Theorem is much the same as for the circulation form of Green's Theorem: Under the proper conditions, the accumulated rotation of the vector field over the surface S (as given by the normal component of the curl) equals the net circulation on the boundary of S.

Ex1: Let C be the intersection of the paraboloid $z = 4 - x^2 - y^2$ and xy-plane. Verify Stokes' Theorem for the vector field $\vec{F}(x, y, z) = 2z\vec{i} + 3x\vec{j} + 5y\vec{k}$ if the paraboloid is oriented upward.

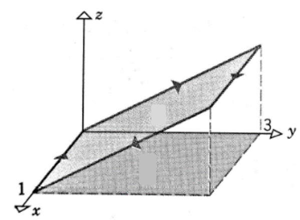


Cool note: If S_1 and S_2 are surfaces with the same positively oriented boundary *C*, then for any vector field $\vec{F}$ that satisfies the hypotheses of Stokes' Theorem, we have that:

$$\iint_{S_1} \left(\nabla \times \vec{F}\right) \cdot \vec{n}\, dS = \oint_C \vec{F} \cdot d\vec{r} = \iint_{S_2} \left(\nabla \times \vec{F}\right) \cdot \vec{n}\, dS$$

This implies that the parabolic surface in Ex1 can be replaced by the upper hemisphere of radius 2 (with upward orientation), or even by the circular region $x^2 + y^2 = 4$ (with upward orientation), without altering the value of the surface integral, since they all have the same oriented boundary C in the xy-plane.

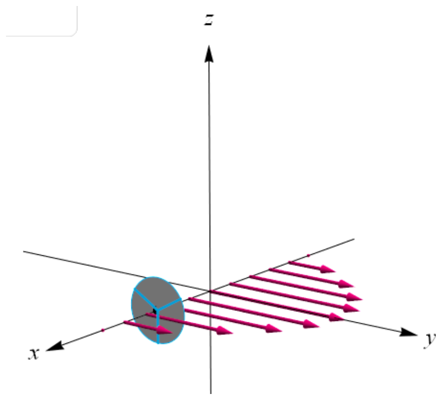
<u>Ex2</u>: Find $\oint_C \vec{F} \cdot d\vec{r}$ if C is the rectangle in the plane $z = y$ with given orientation and $\vec{F}(x, y, z) = x^2\vec{i} + 4xy^3\vec{j} + y^2x\vec{k}$



<u>Ex3</u>: Use Stokes' Theorem to evaluate $\iint_S \text{curl}\vec{F}\cdot d\vec{S}$ for $\vec{F}(x,y,z)=\tan^{-1}\left(x^2yz^2\right)\vec{i}+x^2y\vec{j}+x^2z^2\vec{k}$ where S is the cone $x=\sqrt{y^2+z^2}$, $0\le x\le 2$, oriented in the direction of the positive x-axis.

<u>Ex4</u>: Consider the velocity field $\vec{v} = \langle 0, 1-x^2, 0 \rangle$ for $-1 \le x \le 1$ and $-1 \le z \le 1$ which represents a horizontal flow in the *y*-direction.

a.) Suppose you place a paddle wheel at the point $P\left(\frac{1}{2}, 0, 0\right)$. Using physical arguments, in which of the coordinate directions should the axis of the wheel point in order for the wheel to spin? In which direction does it spin? What happens if you place the wheel at $Q\left(-\frac{1}{2}, 0, 0\right)$?



b.) Compute and graph the curl of $\vec{v}$ and provide an interpretation.

Interpreting curl: As the previous example showed, The direction of $\operatorname{curl} \vec{F} = \nabla \times \vec{F}$ at *P* is the direction in which you should orient the axis of a paddle wheel to obtain the maximum angular speed.