Line Integrals

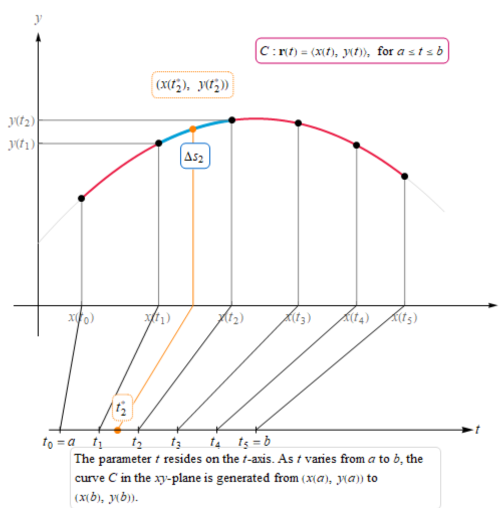
Objective:

1. Scalar line integrals in two and three dimensions
2. Line integrals across vector fields (work)
3. Circulation and flux

I. Scalar line integrals in two and three dimensions

In previous chapters we considered three kinds of integrals in rectangular coordinates: single integrals over intervals, double integrals over two-dimensional regions, and triple integrals over three-dimensional regions. In this section we shall discuss line integrals, which are integrals over curves in two or three-dimensional space. Integrals over curves were invented/discovered in the early 19^{th} century to solve problems involving a variety of things such as fluid flow, force, electricity, and magnetism. The application that is easiest to visualize is the surface area of the curtain "under" a surface and along a curve *C* on the *xy*-plane.

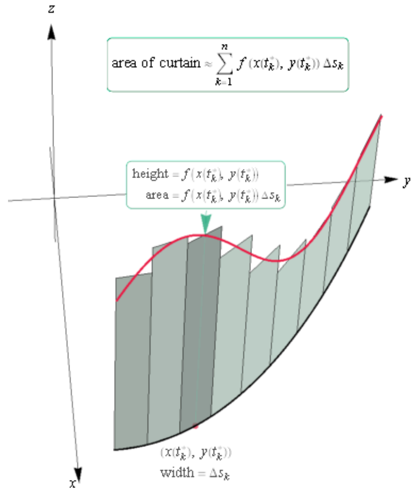
In order to unpack this surface area, we will need to parameterize the curve *C* which requires that we distinguish between the parameter *t* (not visible on the graph) and the arclength *s* (visible).



With this in mind, we can define and calculate the scalar line integral in the plane which represents that area of the curtain "under" *f* and above the *xy*-plane along the curve *C*.

<u>Definition</u>: Scalar Line Integral in the Plane

> Suppose the scalar-valued function *f* is defined on a region containing the smooth curve *C* given by $\vec{r}(t) = \langle x(t), y(t) \rangle$ for $a \le t \le b$.
>
> The line integral of *f* over *C* is $\int_C f\left(x(t), y(t)\right) ds = \lim_{\Delta t \to 0} \sum_{k=1}^{n} f\left(x(t_k^*), y(t_k^*)\right) \Delta s_k$ provided this limit exists over all partitions of [*a*,*b*]. When the limit exists, *f* is said to be integrable on *C*.



The key to applying this definition is finding a formula for ds. To do this, recall that if a space curve is parameterized by $\vec{r}(t)$, then the cumulative arclength of *C* over the interval $[a,t]$ is $s(t)=\int_a^t |\vec{r}\,'(u)|\,du$ which can be differentiated to reveal that $s'(t)=|\vec{r}\,'(t)|$ or using Leibniz notation $ds=|\vec{r}\,'(t)|\,dt$

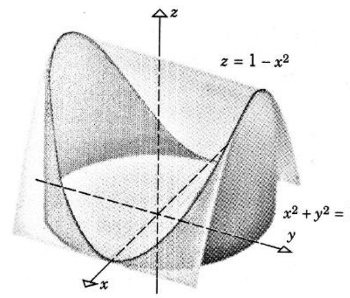
The punchline: To evaluate a scalar line integral "under" *f* and along the path *C* parameterized by $\vec{r}(t)$ with $a \le t \le b$, we use the formulas:

$$\begin{aligned}\text{in } \mathbb{R}^2: \int_C f\,ds &= \int_a^b f\left(x(t),y(t)\right)|\vec{r}\,'(t)|\,dt \\ &= \int_a^b f\left(x(t),y(t)\right)\sqrt{\left(x'(t)\right)^2+\left(y'(t)\right)^2}\,dt\end{aligned}$$

and

$$\begin{aligned}\text{in } \mathbb{R}^3: \int_C f\,ds &= \int_a^b f\left(x(t),y(t),z(t)\right)|\vec{r}\,'(t)|\,dt \\ &= \int_a^b f\left(x(t),y(t),z(t)\right)\sqrt{\left(x'(t)\right)^2+\left(y'(t)\right)^2+\left(z'(t)\right)^2}\,dt\end{aligned}$$

Ex1: Find the area of the surface extending upward from the circle $x^2 + y^2 = 1$ to the parabolic cylinder $z = 1 - x^2$.



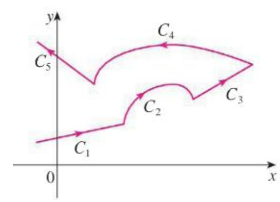
Important notes:

- The procedure for evaluating the line integral $\int_C f\,ds$ (Formulas are given in two dimensions. The three dimensional versions are analogous).
  - Find a parametric description of C in the form $\vec{r}(t)=\langle x(t),y(t)\rangle$ on $a\le t\le b$
  - Compute $|\vec{r}\,'(t)|=\sqrt{(x'(t))^2+(y'(t))^2}$ which we need to find ds.
  - Make substitutions for x and y in the integrand and evaluate an ordinary integral: $\int_C f\,ds=\int f(x(t),y(t))\,|\vec{r}\,'(t)|\,dt$
- The value of the line integral doesn't depend on the parametrization of the curve, provided that the curve is traversed exactly once as t increases from a to b.

- Ordinary single integrals are a special case of the line integral where C is the line segment joining $(a,0)$ and $(b,0)$ with parametric equations $x=x \quad y=0 \quad a\le x\le b$. In this case the line integral formula simplifies from $\int_C f(x,y)\,ds$ to $\int_a^b f(x,0)\,dx$

- If C is a **piecewise-smooth curve**, then we find the line integral for each piece and add them to get the line integral over C.

$$\int_C f(x,y)\,ds=\int_{C_1} f(x,y)\,ds+\int_{C_2} f(x,y)\,ds+\cdots+\int_{C_n} f(x,y)\,ds$$



- Any physical interpretation of a line integral depends on the physical interpretation of the function f. For example, if f represents the density of a thin wire shaped like C, then the total mass is the line integral of f over C.

- When setting up a line integral, the most difficult step is parameterizing the curve C. Three common parameterizations are:
  - A line segment that starts with $\vec{r_0}$ and ends in $\vec{r_1}$ is given by: $\vec{r}(t)=(1-t)\vec{r_0}+t\vec{r_1} \quad 0\le t\le 1$
  - A circle $(0\le t\le 2\pi)$ or a semicircle $(0\le t\le \pi)$ centered at $(x,y)=(a,b)$ with radius r is given by $x=a+r\cos t$ and $y=b+r\sin t$ for counter clockwise movement.
  - A curve that can be represented by a function $y=f(x)$ on $a\le x\le b$. In this case we let $x=t$ which makes $y=f(t)$. So $\vec{r}(t)=\langle t,f(t)\rangle$ on $a\le t\le b$.

II. Line integrals across vector fields (work)

This is what we have learned so far about work:

(review): Work Done by a Constant Force:

- If an object is moved a distance d in the direction of an applied constant force F, then the work W done by the force is defined as $W = Fd$
- The work done by a constant force $\vec{F}$ that moves an object from P to Q (creating displacement vector $\vec{D}$) can be calculated by $W = \vec{F} \cdot \vec{D}$

(review): Work Done by a Variable Force:

- If an object moves along a straight line from a to b, subject to a continuously varying force (not constant) $f(x)$, we define the work as $W = \int_a^b f(x)\,dx$

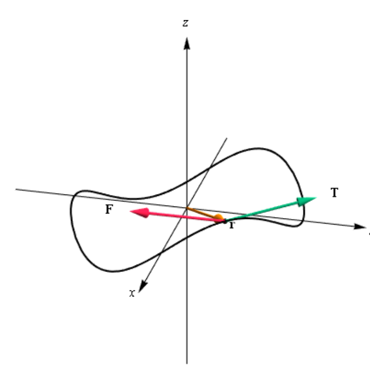
(new): Work required to move a particle through a vector field:

- If an object moves along a smooth curve C, through a continuous force field $\vec{F} = f(x,y)\vec{i} + g(x,y)\vec{j}$ defined on $\mathbb{R}^2$, then the work done is obtained by integrating the tangential component of force along the curve so $W = \int_C \vec{F} \cdot \vec{T}\,ds$ where $\vec{T}(x,y)$ is the unit tangent vector at the point (x,y) on C.

  At first glance this looks complicated, but work is actually straight forward to implement. Recall that if the curve is given by the vector equation $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j}$, then $\vec{T}(t) = \dfrac{\vec{r}\,'(t)}{\left|\vec{r}\,'(t)\right|}$.

  This means that we calculate work using the formula:

$$W = \int_C \vec{F} \cdot \vec{T}\,ds = \int_C \vec{F} \cdot \underbrace{\frac{\vec{r}\,'(t)}{\left|\vec{r}\,'(t)\right|}}_{\vec{T}} \underbrace{\left|\vec{r}\,'(t)\right| dt}_{ds} = \int_C \vec{F} \cdot d\vec{r}$$



  Or simply $\text{work} = \int_C \vec{F} \cdot d\vec{r}$

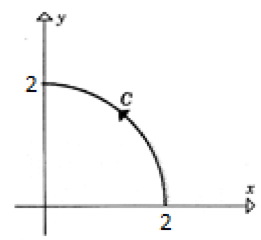
  The definition in three dimensions is analogous.

Ex2: Find the work done by the force field $\vec{F} = x^3 y\vec{i} + (x - y)\vec{j}$ on a particle that moves along the parabola $y = x^2$ from $(-2, 4)$ to $(1,1)$.

One common way to write the work formula is to reference *x*, and *y* rather than *r*. To do this, recall that $\vec{r}(t)=\langle x(t),y(t)\rangle$. This means that $\frac{d\vec{r}}{dt}=\vec{r}\,'(t)=\langle x'(t),y'(t)\rangle$ and the work formula is: $\text{work}=\int_C \vec{F}\cdot d\vec{r}=\int_C \underbrace{\langle f(x,y),g(x,y)\rangle}_{\vec{F}}\cdot\underbrace{\langle x'(t),y'(t)\rangle dt}_{d\vec{r}}=\int_C f(x,y)\underbrace{dx}_{x'(t)dt}+g(x,y)\underbrace{dy}_{y'(t)dt}$ or simply

$$\text{work}=\int_C \vec{F}\cdot d\vec{r}=\int_C f\,dx+g\,dy$$

<u>Ex3</u>: Evaluate the $\text{work}=\int_C 2xy\,dx+\left(x^2+y^2\right)dy$ required to move a particle along the circular arc C.



In general, the value of a line integral (and work) depends on three things:

- The location of the endpoints of the curve.
- The length and shape of the path (later will we learn conditions that cause the integral to be independent of path)
- The direction or orientation of the curve.

If $-C$ denotes the curve consisting of the same points as C but with the opposite orientation (from initial point B to terminal point A in Figure 8), then we have

$$\int_{-C} f(x, y)\, dx = -\int_C f(x, y)\, dx \qquad \int_{-C} f(x, y)\, dy = -\int_C f(x, y)\, dy$$

But if we integrate with respect to arc length, the value of the line integral does *not* change when we reverse the orientation of the curve:

$$\int_{-C} f(x, y)\, ds = \int_C f(x, y)\, ds$$

This is because Δs_i is always positive, whereas Δx_i and Δy_i change sign when we reverse the orientation of C.

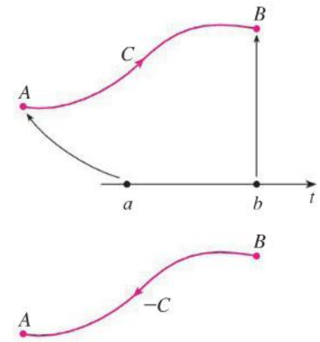
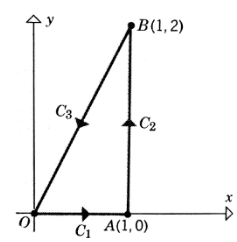


FIGURE 8

Ex4: Evaluate the $\text{work} = \int_C x^2 y\, dx + x\, dy$ for a particle to move in a counterclockwise direction around the triangular path given.



Ex5: Evaluate $I = \int_C \left(xy + z^3 \right) ds$ where C is the portion of the helix given by the parametric equations:

$x = \cos t \quad y = \sin t \quad z = t \quad 0 \le t \le \pi$

Ex6: Find the mass of a thin wire if the density function is $\rho(x, y, z) = kz$ with $k > 0$ and shaped in the form of a helix with parametric equations $x = \cos t \quad y = \sin t \quad z = 2t \quad 0 \le t \le \pi$.

Ex7: Evaluate $\text{work} = \int_C yz\,dx + xz\,dy + xy\,dz$ where C is $\vec{r}(t) = t\vec{i} + t^2\vec{j} + t^3\vec{k}$ on $0 \le t \le 1$.

III. Circulation and flux

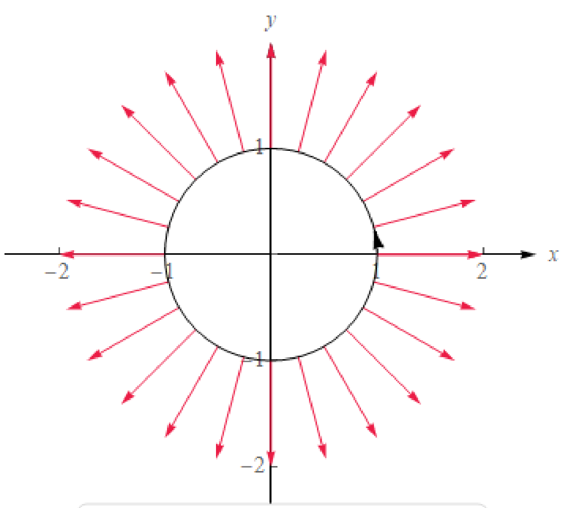
A number of the forthcoming theorems/results rely on the idea of a closed path (or closed curve). A closed curve is one whose initial and terminal points are the same.

In this case, we give the work to move an object around a closed path a new name: circulation.

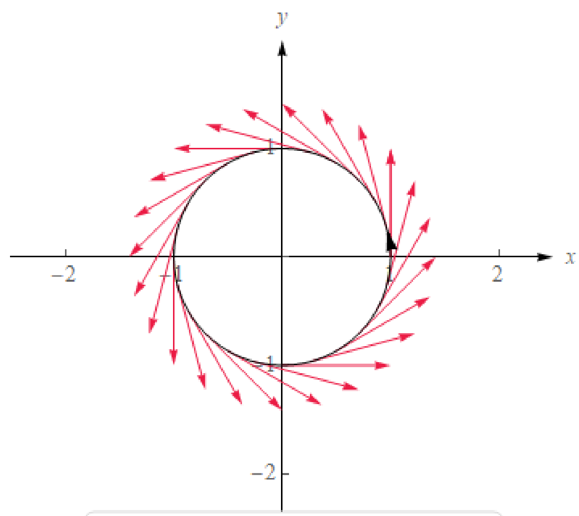
Definition: Circulation

> Let $\vec{F}$ be a continuous vector field on a region *D* of $\mathbb{R}^3$ and let *C* be a closed smooth oriented curve in *D*. The circulation of $\vec{F}$ is $\text{circulation} = \text{work} = \oint_C \vec{F} \cdot \vec{T}\, ds$ where $\oint$ represents the line integral around a closed path.

Ex8: Let *C* be the unit circle with counterclockwise orientation. Find the circulation on *C* for the radial flow field $\vec{F} = \langle x, y \rangle$ and the rotation flow field $\vec{F} = \langle -y, x \rangle$



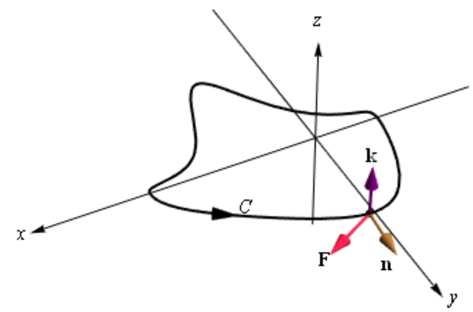
On the unit circle, $\mathbf{F} = \langle x, y \rangle$ is orthogonal to C and has zero circulation on C.



On the unit circle, $\mathbf{F} = \langle -y, x \rangle$ is tangent to C and has positive circulation on C.

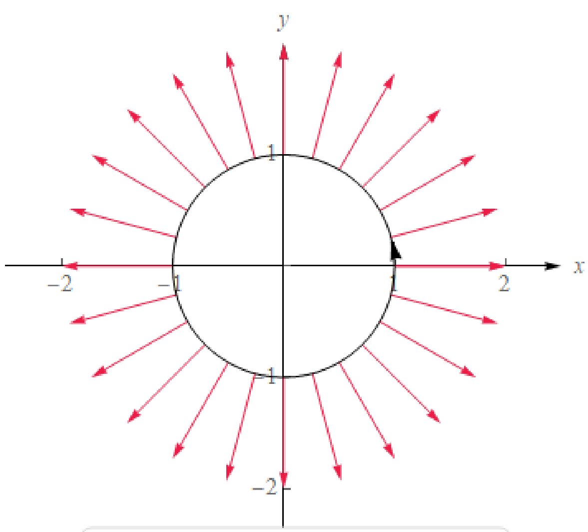
While circulation is a measure of the tangential component of field along a path, flux is a measure of the normal component of the field along a path.

In two dimensions, this will be $\text{flux} = \int_C \vec{F} \cdot \vec{n}\, ds$ where $\vec{n} = \vec{T} \times \vec{k}$ ($\vec{k}$ being the unit vector in the z direction).

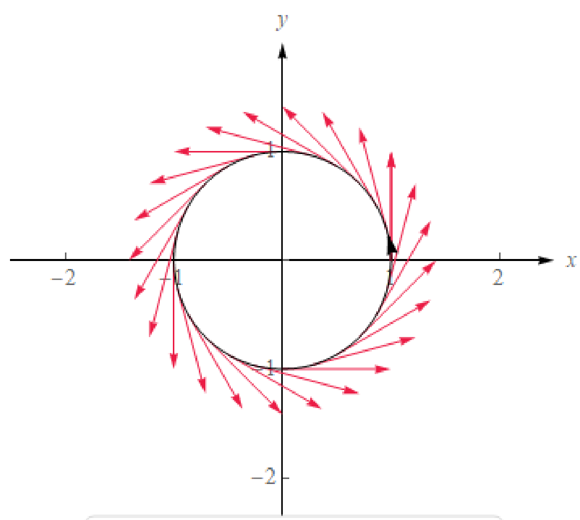


F points outward on C and gives a positive contribution to flux.

While we may not yet be able to calculate the flux, we already know enough to compare the flux and circulation in radial and rotational flows.



On the unit circle, $\mathbf{F} = \langle x, y \rangle$ is orthogonal to C and has positive outward flux on C.



On the unit circle, $\mathbf{F} = \langle -y, x \rangle$ is tangent to C and has zero outward flux on C.

Ex9: Derive the (computationally nice) formula for flux in two dimensions where $\vec{F} = \langle f, g \rangle$ represents a continuous vector field traversed by path *C*: $\vec{r}(t) = \langle x(t), y(t) \rangle$ for $a \le t \le b$.