IN THIS CHAPTER WE STUDY the calculus of vector fields. (These are functions that assign vectors to points in space.) In particular we define line integrals (which can be used to find the work done by a force field in moving an object along a curve). Then we define surface integrals (which can be used to find the rate of fluid flow across a surface). The connections between these new types of integrals and the single, double, and triple integrals that we have already met are given by the higher-dimensional versions of the Fundamental Theorem of Calculus: Green's Theorem, Stokes' Theorem, and the Divergence Theorem.

Vector Fields

Objective:

1. Vector fields and examples
2. Gradient Fields

1. Vector Fields

In this section we consider functions called **vector fields** that associate vectors with points in two (or three) dimensions. That means they have domain as a set of points in $\mathbb{R}^2$ (or $\mathbb{R}^3$) and range as a set of vectors in V_2 (or V_3). These functions play an important role in the study of fluid flow, gravitational force fields, electromagnetic force fields and a wide range of other applied problems. We generally use $\vec{F}$ to denote a vector field.

For two dimensions:

$$\vec{F}(x,y) = f(x,y)\vec{i} + g(x,y)\vec{j} = \langle f(x,y), g(x,y) \rangle$$

For three dimensions:

$$\vec{F}(x,y,z) = f(x,y,z)\vec{i} + g(x,y,z)\vec{j} + h(x,y,z)\vec{k} = \langle f(x,y,z), g(x,y,z), h(x,y,z) \rangle$$

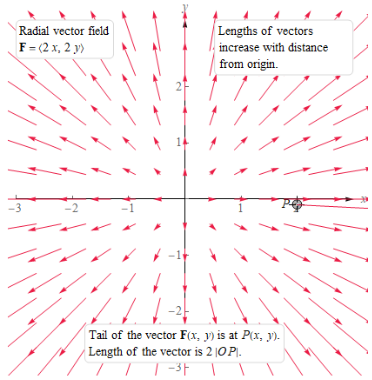
We call $f(x,y,z), g(x,y,z)$ and $h(x,y,z)$ **component functions**.

As with the vector functions in the previous class, we can define continuity of vector fields and show that $\vec{F}$ is **continuous** if and only if its component functions are continuous.

What follows are a number of examples with some commentary.

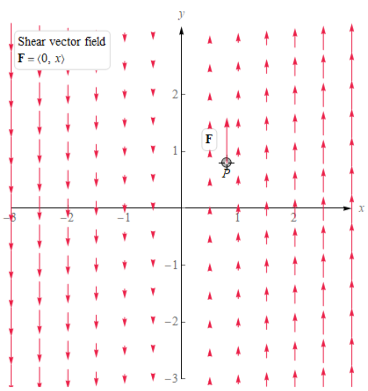
Ex1: The radial vector field $\vec{F} = \langle 2x, 2y \rangle$.

Note: Drawing vectors with their actual length often leads to cluttered pictures of vector fields. For this reason, most vector fields are illustrated with proportional scaling: All vectors are multiplied by a scalar chosen to make the vector field as understandable as possible.

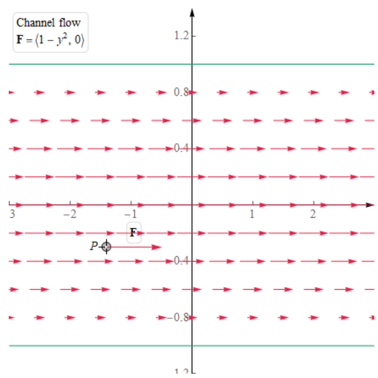


Ex2: Three examples

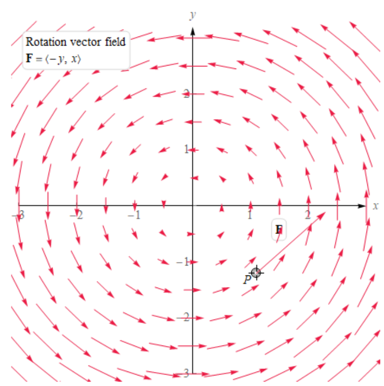
Shear Field
$\vec{F} = \langle 0, x \rangle$



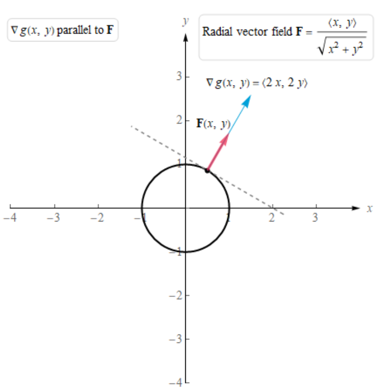
Channel Flow
$\vec{F} = \langle 1 - y^2, 0 \rangle$

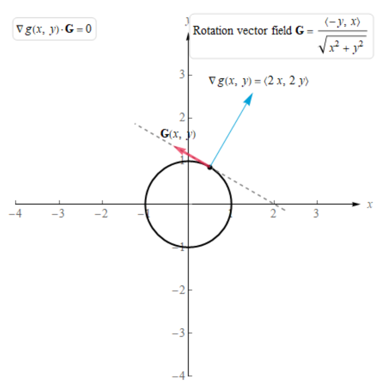


Rotation Field
$\vec{F} = \langle -y, x \rangle$

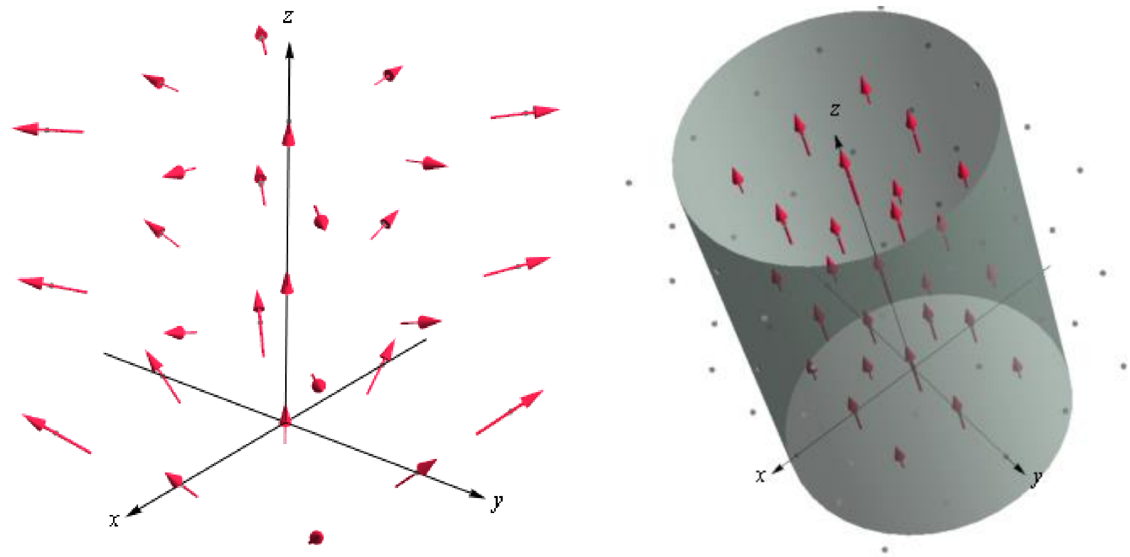


Ex3: Radial and rotation fields





Ex4: Vector fields in three dimensions



2. Gradient Fields

If f is a scalar function of two variables, recall that its gradient ∇f (or grad f) is defined by

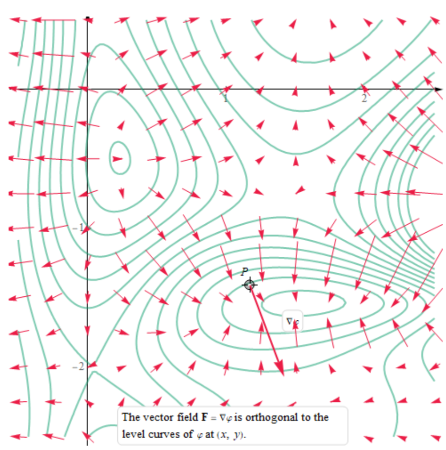
$$\nabla f(x, y) = f_x(x, y)\,\mathbf{i} + f_y(x, y)\,\mathbf{j}$$

Therefore ∇f is really a vector field on $\mathbb{R}^2$ and is called a **gradient vector field**. Likewise, if f is a scalar function of three variables, its gradient is a vector field on $\mathbb{R}^3$ given by

$$\nabla f(x, y, z) = f_x(x, y, z)\,\mathbf{i} + f_y(x, y, z)\,\mathbf{j} + f_z(x, y, z)\,\mathbf{k}$$

At each point in a gradient field where the gradient is nonzero, the vector points in the direction in which the rate of increase of f is maximum.

Ex6: Gradient fields are orthogonal to level curves.



Definition: Gradient Fields and Potential Functions

Let φ be differentiable on a region of $\mathbb{R}^2$ or $\mathbb{R}^3$. The vector field $\vec{F} = \nabla\varphi$ is a gradient field, and the function φ is a potential function for $\vec{F}$.

Note: A potential function plays the role of an antiderivative of a vector field: Derivatives of the potential function produce the vector field. If φ is a potential function for a gradient field, then φ + C is also a potential function for that gradient field, for any constant C.

Definition: Equipotential Curves and Surfaces

The level curves of a potential function are called equipotential curves (curves on which the potential function is constant).

Because the equipotential curves are level curves of φ, the vector field $\vec{F} = \nabla\varphi$ is everywhere orthogonal to the equipotential curves. Therefore, the vector field is visualized by drawing continuous flow curves or streamlines that are everywhere orthogonal to the equipotential curves.