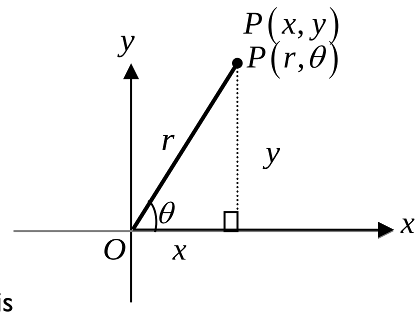
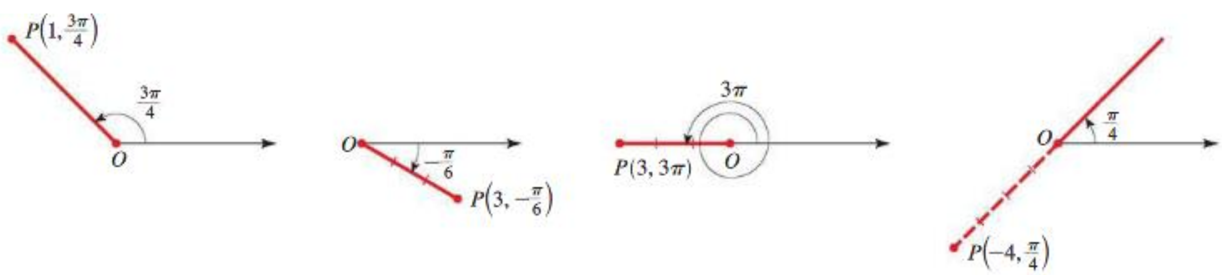
Polar Coordinates

We've been using the Cartesian (rectangular) coordinate system to graph. Another useful coordinate system is the **polar coordinate system** which uses distance and direction to specify the location of a point in the plane. The system is based on a point, called the **pole** (or origin) and a ray drawn in the direction of the x-axis, called the **polar axis**.



r is the distance from O to P.
θ is the angle between the polar axis and the segment $\overline{OP}$.

θ is positive if measures in a counterclockwise direction from the polar axis or negative if measured in a clockwise direction. If r is negative, then P is the point that is $|r|$ units from the pole in the direction opposite to that given by θ.



<u>**Ex1**</u>: Plot the given points.

a) $\left(2, \frac{2\pi}{3}\right)$

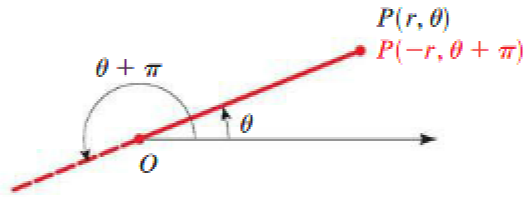
b) $\left(1, -\frac{\pi}{4}\right)$

c) $(4, 3\pi)$

d) $\left(-3, \frac{\pi}{6}\right)$

<u>Note</u>: The coordinates (r,θ) and $(-r,\theta+\pi)$ represent the same point. Also keep in mind that because of the coterminal angles, each point has infinitely many representations. $P(r,\theta)$ is the same as:

$P(r,\theta+2n\pi)$ and $P(-r,\theta+(2n+1)\pi)$



<u>Ex2</u>: Graph $Q\left(1,\frac{5\pi}{4}\right)$. Find two other polar coordinate representations with $r>0$, and two with $r<0$.

RELATIONSHIP BETWEEN POLAR AND RECTANGULAR COORDINATES

1. To change from polar to rectangular coordinates, use the formulas

$$x = r\cos\theta \qquad \text{and} \qquad y = r\sin\theta$$

2. To change from rectangular to polar coordinates, use the formulas

$$r^2 = x^2 + y^2 \qquad \text{and} \qquad \tan\theta = \frac{y}{x} \quad (x \neq 0)$$

<u>Note</u>: These equations do not uniquely determine r or θ. Check to make sure you are in the correct quadrant.

<u>Ex3</u>: Find the rectangular coordinates of the point $\left(-4,\frac{3\pi}{4}\right)$.

Ex4: Find the polar coordinates of the point $\left(-1,-\sqrt{3}\right)$.

Ex5: Express the following equations in polar coordinates.

a) $x=2$

b) $x^2+y^2=9$

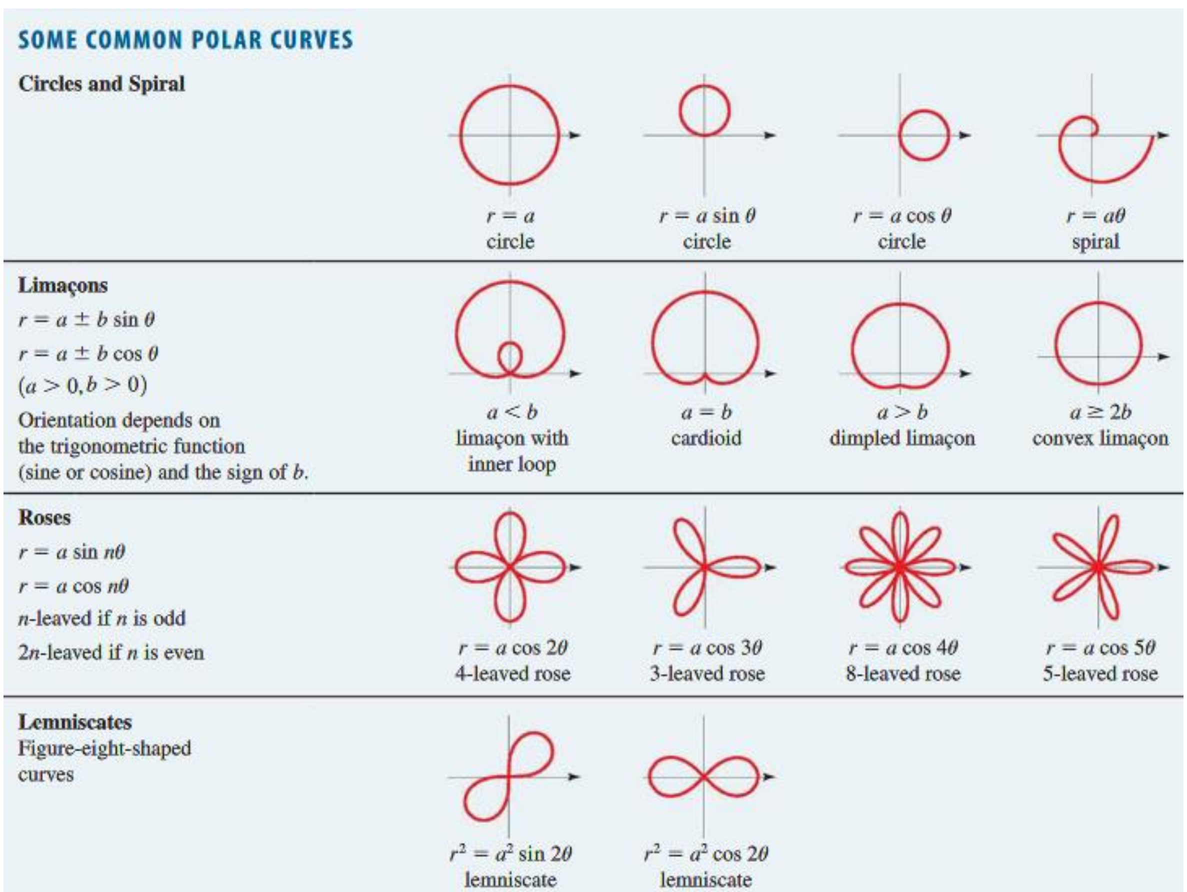
Ex6: Express the following equations in rectangular coordinates.

a) $r=\dfrac{4}{1+\sin\theta}$

b) $\tan\theta = 1$

❖ ***Polar Curves***

The **graph of a polar equation** $r = f(\theta)$, or more generally $F(r, \theta) = 0$, consists of all points P that have at least one polar representation (r, θ) whose coordinates satisfy the equation.



❖ ***Tangents to Polar Curves***

To find a tangent line to a polar curve $r = f(\theta)$, we regard θ as a parameter and write its parametric equations as

$$x = r\cos\theta = f(\theta)\cos\theta \qquad y = r\sin\theta = f(\theta)\sin\theta$$

Then using the method for finding slopes of parametric curves (from Calculus III) and the Product Rule, we have

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{d}{d\theta}(r\sin\theta)}{\frac{d}{d\theta}(r\cos\theta)} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

Note: $\frac{dy}{dx} \neq \frac{df}{d\theta}$

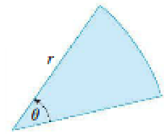
Notice that if we are looking for tangent lines at the pole, then $r = 0$ and the equation above simplifies to:

$$\frac{dy}{dx} = \tan\theta \qquad \text{if} \quad \frac{dr}{d\theta} \neq 0$$

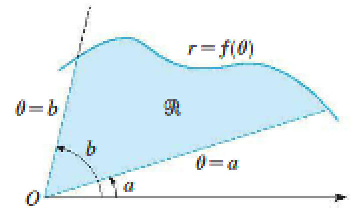
<u>**Ex9**</u>: Find the slope of the tangent line to $r = 1 - \cos\theta$ at θ.

❖ *Area in Polar Coordinates*

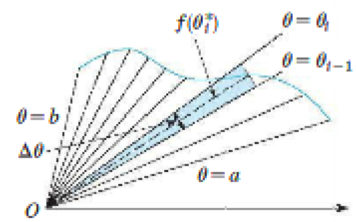
Recall the formula for the area of a sector: $A = \frac{1}{2} r^2 \theta$



Now suppose that we want to find the area of the shaded region on the given graph where f is positive and continuous and $0 < b - a \le 2\pi$.



Divide $[a, b]$ into subintervals of equal width $\Delta\theta$. Then:



$$\Delta A_i \approx \frac{1}{2}\left[f\left(\theta_i^*\right)\right]^2 \Delta\theta$$

$$A \approx \sum_{i=1}^{n} \frac{1}{2}\left[f\left(\theta_i^*\right)\right]^2 \Delta\theta$$

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{2}\left[f\left(\theta_i^*\right)\right]^2 \Delta\theta = \int_a^b \frac{1}{2}\left[f\left(\theta\right)\right]^2 d\theta$$

This is often written as:

$$A = \int_a^b \frac{1}{2} r^2 \, d\theta$$

Ex10: Find the area of the region in the plane enclosed by the polar rose $r = 2\cos(2\theta)$.

<u>Ex11</u>: Find the area of the region that lies inside the circle $r = 1$ and outside the cardioid $r = 1 + \cos\theta$.

❖ ***Length of the Polar Curves***

To find the length of a polar curve $r = f(\theta)$, $a \leqslant \theta \leqslant b$, we regard θ as a parameter and write the parametric equations of the curve as

$$x = r\cos\theta = f(\theta)\cos\theta \qquad y = r\sin\theta = f(\theta)\sin\theta$$

Using the Product Rule and differentiating with respect to θ, we obtain

$$\frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta \qquad \frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta$$

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = \left(\frac{dr}{d\theta}\right)^2 \cos^2\theta - 2r\frac{dr}{d\theta}\cos\theta\ \sin\theta + r^2\sin^2\theta$$

$$+ \left(\frac{dr}{d\theta}\right)^2 \sin^2\theta + 2r\frac{dr}{d\theta}\sin\theta\ \cos\theta + r^2\cos^2\theta$$

$$= \left(\frac{dr}{d\theta}\right)^2 + r^2$$

Assuming that f' is continuous, we can use what we learned in calculus III and find:

$$L = \int_a^b \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

Therefore the length of a curve with polar equation $r = f(\theta)$, $a \leqslant \theta \leqslant b$, is

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

<u>Ex12</u>: Find the length of the cardioid $r = 1 + \cos\theta$.