Strategy for Testing Series

Let's do a brief review of THE LIST of all the tests we have introduced for determining convergence and how to decide which test to use.

THE LIST: (1.) The geometric series converges when $|r|<1$. (2.) The harmonic series diverges. (3.) Telescoping series. (4.) The integral test. (5.) The test for divergence. (6.) The p-series converges for $p>1$. (7.) The comparison test (weak). (8.) The limit comparison test (stronger). (9.) The alternating harmonic series converges. (10.) The alternating series test. (11.) The ratio test. (12.) The root test.

Let us break down THE LIST by beginning with the four known or famous series:

- Geometric series: $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$ $\begin{cases} \text{converges} & \text{if} \quad |r|<1 \\ \text{diverges} & \text{if} \quad |r|\ge 1 \end{cases}$
- P-series: $\sum_{n=1}^{\infty} \frac{1}{n^p}$ $\begin{cases} \text{converges} & \text{if} \quad p>1 \\ \text{diverges} & \text{if} \quad p\le 1 \end{cases}$
- The Harmonic Series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges
- The Alternating Harmonic Series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges (conditionally) to $\ln(2)$

What if a series is given and you need to determine whether it diverges or converges (conditionally or absolutely). Let $\sum_{n=1}^{\infty} a_n$ be given.

We will organize our series using the following chart

	EASY	HARD
WEAK		
STRONG		

__**For Positive Series:**__

- **The Test for Divergence (easy-weak):**

$$\lim_{n\to\infty} a_n = \begin{cases} 0 & \text{then} \quad \text{inconclusive!} \\ \neq 0 & \text{then} \quad \sum_{n=1}^{\infty} a_n \quad \text{diverges} \end{cases}$$

  Pro-tip: Always check this test first.

- **The Direct Comparison Test (hard-strong):**

  If $a_n \le b_n$ and $\sum_{n=1}^{\infty} b_n$ converges then $\sum_{n=1}^{\infty} a_n$ converges

  If $a_n \ge b_n$ and $\sum_{n=1}^{\infty} b_n$ diverges then $\sum_{n=1}^{\infty} a_n$ diverges

In table form, if $\sum b_n$ is known and …

	BIGGER	SMALLER
CONVERGES	$\sum a_n$ converges	test inconclusive
DIVERGES	test inconclusive	$\sum a_n$ diverges

<u>Pro-tip</u>: Consider whether dropping terms in the numerator or denominator gives a series that we know converges or diverges, $\sum_{n=1}^{\infty} b_n$.

- **The Limit Comparison Test (easy -strong):**
  - If $\lim_{n\to\infty} \frac{a_n}{b_n} \begin{cases} >0 \\ finite \end{cases}$, then either $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge or both diverge.
  - If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.
  - If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

In table form, if $\sum b_n$ is known and …

CONVERGES	$\sum a_n$ converges
DIVERGES	$\sum a_n$ diverges

<u>Pro-tip</u>: Consider whether dropping terms in the numerator or denominator gives a series that we know converges or diverges, $\sum_{n=1}^{\infty} b_n$.

- **The Integral Test (hard-strong):**

If $a_n = f(n)$ is decreasing, positive and continuous,

$$\int_1^\infty f(x)\,dx \quad \begin{cases} \text{converges} & \text{then} & \sum_{n=1}^{\infty} a_n & \text{converges} \\ \text{diverges} & \text{then} & \sum_{n=1}^{\infty} a_n & \text{diverges} \end{cases}$$

Pro-tip: Best to use when the other tests fail.

- **Telescoping Series (hard- weak):**

Use this when the terms can be written as a difference where consecutive-ish terms cancel. This often will require the use of partial fractions decomposition, log rules, or trig identities.

Pro-tip: Best to use when the other tests fail.

<u>For series that are not (necessarily) positive series:</u>

- **Alternating Series Test (easy-weak):**

For series of the form $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^{n-1} b_n$ where $b_n \geq 0$

$$\text{If} \quad \begin{cases} b_{n+1} \leq b_n \\ \lim_{n\to\infty} b_n = 0 \end{cases} \quad \text{then} \quad \sum_{n=1}^{\infty} a_n \quad \text{converges (either absolutely or conditionally).}$$

Pro-tip: It is the last note that makes this weak. If convergent using the Alternating Series Test than you still must determine if you have conditional or absolute convergence.

Reminder: <u>Absolute Convergence</u>: If $\sum_{n=1}^{\infty} a_n$ has some negative terms (maybe alternating or not), then consider $\sum_{n=1}^{\infty} |a_n|$. If $\sum_{n=1}^{\infty} |a_n|$ converges, $\sum_{n=1}^{\infty} a_n$ is converging absolutely.

- **The Ratio Test (easy -strong):**

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \quad \begin{cases} <1 & \text{then} & \text{absolutely-convergent} \Rightarrow \text{convergent} \\ >1 & \text{then} & \text{divergent} \\ 1 & \text{then} & \text{inconclusive} \end{cases}$$

Pro-tip: Best to use when there is a factorial or powers of *n*. <u>Don't use with p-series</u>.

- **The Root Test (easy - weak):**

$$\lim_{n\to\infty}\sqrt[n]{|a_n|}\begin{cases} <1 & \text{then} & \text{absolutely-convergent} \Rightarrow \text{convergent} \\ >1 & \text{then} & \text{divergent} \\ 1 & \text{then} & \text{inconclusive} \end{cases}$$

Pro-tip: Best to use when the terms have power *n* (but no other complicating terms). Don't use with p-series.

In short:

1. **The Test for Divergence**: Unless $a_n \to 0$, the series diverges.
2. **Geometric series**: $\sum ar^n$ converges if $|r|<1$; otherwise it diverges.
3. **P-series**: $\sum \frac{1}{n^p}$ converges if $p>1$; otherwise it diverges.
4. **Series with nonnegative terms**: Try the Integral Test, Ratio Test or Root Test. Try comparing to a known series with the Comparison Test or the Limit Comparison Test.
5. **Series with some negative terms**: Does $\sum |a_n|$ converges? If yes, so does $\sum a_n$ since absolute convergence implies convergence.
6. **Alternating series**: To show conditional convergence, use the Alternating Series Test $\sum a_n$ converges if $\sum |a_n|$ is decreasing and $|a_n| \to 0$. To show absolute convergence, use the Ratio or Root Test.

Summary of how to prioritize your choice of tests

	EASY	HARD
WEAK	• Test for divergence • Alternating series test • Harmonic series • Alternating harmonic series • Geometric series • P-series • Root test	• Telescoping series
STRONG	• Limit comparison test • Ratio test	• Comparison test • Integral test