The Comparison Tests

The Comparison Test Suppose that $\Sigma\, a_n$ and $\Sigma\, b_n$ are series with positive terms.

(i) If $\Sigma\, b_n$ is convergent and $a_n \leqslant b_n$ for all n, then $\Sigma\, a_n$ is also convergent.

(ii) If $\Sigma\, b_n$ is divergent and $a_n \geqslant b_n$ for all n, then $\Sigma\, a_n$ is also divergent.

In using the Comparison Test we must, of course, have some known series $\Sigma\, b_n$ for the purpose of comparison. Most of the time we use one of these series:

- A p-series [$\Sigma\, 1/n^p$ converges if $p > 1$ and diverges if $p \leqslant 1$;
- A geometric series [$\Sigma\, ar^{n-1}$ converges if $|r| < 1$ and diverges if $|r| \geqslant 1$;

<u>Ex1</u>: Determine the convergence of the following.

a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}\,3^n}$

b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\left(n^2+3\right)^{1/3}}$

Visualize the power and limitations of the comparison test as follows:

NOTE 1 Although the condition $a_n \leqslant b_n$ or $a_n \geqslant b_n$ in the Comparison Test is given for all n, we need verify only that it holds for $n \geqslant N$, where N is some fixed integer, because the convergence of a series is not affected by a finite number of terms.

<u>Ex2</u>: Determine the convergence of the following.

$$\sum_{n=1}^{\infty} \frac{n^2}{n^4 + n - 10}$$

THE LIST: (1.) The geometric series converges when $|r| < 1$. (2.) The harmonic series diverges. (3.) Telescoping series. (4.) The integral test. (5.) The test for divergence. (6.) The p-series converges for $p > 1$. (7.) The comparison test (weak).

NOTE 2 The terms of the series being tested must be smaller than those of a convergent series or larger than those of a divergent series. If the terms are larger than the terms of a convergent series or smaller than those of a divergent series, then the Comparison Test doesn't apply.

To be able to answer the last problem we use the following variation of the Direct Comparison test.

> **The Limit Comparison Test** Suppose that $\Sigma\, a_n$ and $\Sigma\, b_n$ are series with positive terms. If
>
> $$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$
>
> where c is a finite number and $c > 0$, then either both series converge or both diverge.

A couple of special cases:

- If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.
- If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

Back to <u>Ex2:</u> $\sum_{n=1}^{\infty} \frac{n^2}{n^4 + n - 10}$

Back to Ex1b: $\sum_{n=1}^{\infty} \frac{1}{\left(n^2+3\right)^{1/3}}$

THE LIST: (1.) The geometric series converges when $|r| < 1$. (2.) The harmonic series diverges. (3.) Telescoping series. (4.) The integral test. (5.) The test for divergence. (6.) The p-series converges for $p > 1$. (7.) The comparison test (weak). (8.) The limit comparison test (stronger)