Sequences

A **sequence** can be thought of as a list of numbers written in a definite order:

$$a_1,\ a_2,\ a_3,\ a_4,\ \ldots,\ a_n, \ldots$$

The number a_1 is called the *first term*, a_2 is the *second term*, and in general a_n is the *nth term*. We will deal exclusively with infinite sequences and so each term a_n will have a successor a_{n+1}.

Notice that for every positive integer n there is a corresponding number a_n and so a sequence can be defined as a function whose domain is the set of positive integers. But we usually write a_n instead of the function notation $f(n)$ for the value of the function at the number n.

NOTATION The sequence $\{a_1, a_2, a_3, \ldots\}$ is also denoted by

$$\{a_n\} \qquad \text{or} \qquad \{a_n\}_{n=1}^{\infty}$$

When a_n is given by a formula we refer to it as the general term of a sequence.

❖ ***Arithmetic Sequences***

A sequence is arithmetic if there is a common difference, d, between two consecutive terms so $a_n = a_{n-1} + d$.

Ex1: 2, 5, 8, ...

a) What is d?

b) What is a_1?

c) What is the 4th term of this sequence?

➢ The nth term of an arithmetic sequence is given by: $a_n = a_1 + (n-1)d$

d) What is the 20th term of this sequence?

❖ ***Geometric Sequences***

A sequence is geometric if there is a common ratio, r, between two consecutive terms so $r = \dfrac{a_n}{a_{n-1}}$ or $a_n = ra_{n-1}$.

Ex2: 3, 6, 12, ...

a) What is r?

b) What is a_1?

c) What is the 4th term of this sequence?

➢ The nth term of a geometric sequence is given by: $a_n = a_1 r^{n-1}$

d) What is the 20th term of this sequence?

Ex3: Determine whether each sequence is arithmetic or geometric. Find the next term.

a) $1, \quad 3, \quad 5, \quad \ldots$

b) $2, \quad 4, \quad 8, \quad \ldots$

c) $6, \quad 3, \quad 1.5, \quad \ldots$

d) $12, \quad 7, \quad 2, \quad -3, \quad \ldots$

e) $3, \quad -30, \quad 300, \quad -3000, \quad \ldots$

Ex4: What is the general term of the sequence $\left\{\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1, \frac{5}{4}, \frac{3}{2}, \ldots\right\}$? How else can you present this sequence?

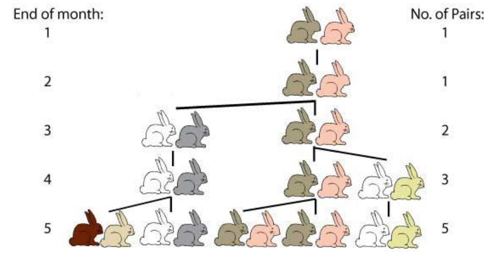
Ex5: Write out the first few terms of the sequence $\left\{(-1)^n \sqrt{n}\right\}_{n=3}^{\infty}$

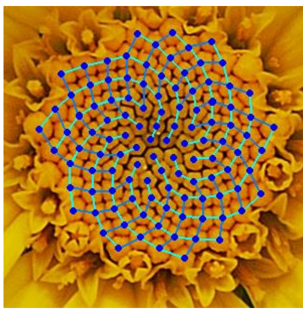
Not all sequences are generated by a formula. For instance the sequence $\{3, 1, 4, 1, 5, 9, 2, 6, \ldots\}$ is the digits of π, and there is no formula for the n^{th} digit of π.

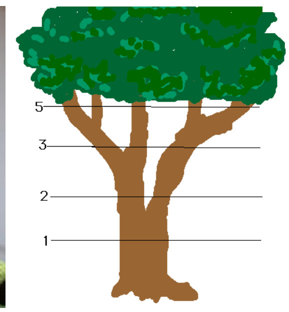
A very famous sequence is the *Fibonacci sequence*: $\{1, 1, 2, 3, 5, 8, 13, 21, \ldots\}$ Do you see the pattern?

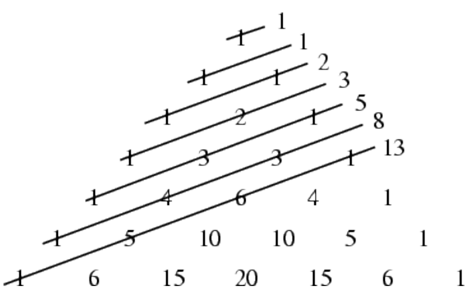
This sequence is defined recursively. This means the first one or two terms may be given and other terms are found by a pattern applied to the preceding terms.

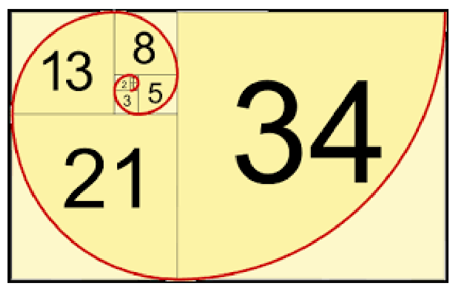
This sequence arose when the 13th-centry Italian mathematician Fibonacci posed a problem concerning the breeding of rabbits. This particular sequence appears in a surprisingly wide variety of situations, particularly in nature. For instance, the number of spiral arms on a sunflower almost always turns out to be a number from this sequence.

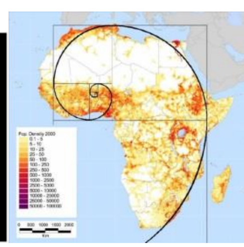


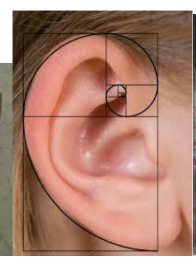












Convergent Sequences

Now consider the sequence $a_n = \dfrac{n}{n+1}$ and picture it two ways:

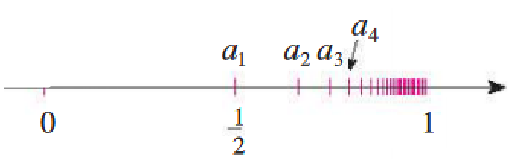


Figure 1

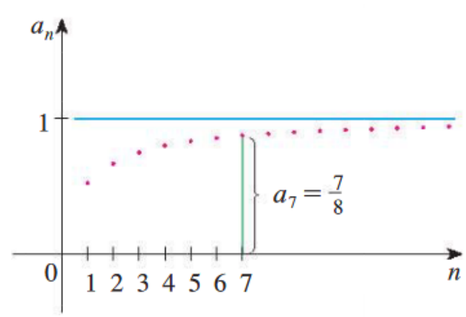


Figure 2

Note that, since a sequence is a function whose domain is the set of positive integers, its graph consists of isolated points with coordinates

$$(1, a_1) \qquad (2, a_2) \qquad (3, a_3) \qquad \ldots \qquad (n, a_n) \qquad \ldots$$

From Figure 1 or Figure 2 it appears that the terms of the sequence $a_n = n/(n+1)$ are approaching 1 as n becomes large. In fact, the difference

$$1 - \frac{n}{n+1} = \frac{1}{n+1}$$

can be made as small as we like by taking n sufficiently large. We indicate this by writing

$$\lim_{n\to\infty} \frac{n}{n+1} = 1$$

In general, the notation

$$\lim_{n\to\infty} a_n = L$$

means that the terms of the sequence $\{a_n\}$ approach L as n becomes large. Notice that the following definition of the limit of a sequence is very similar to the definition of a limit of a function at infinity given in Section 2.6.

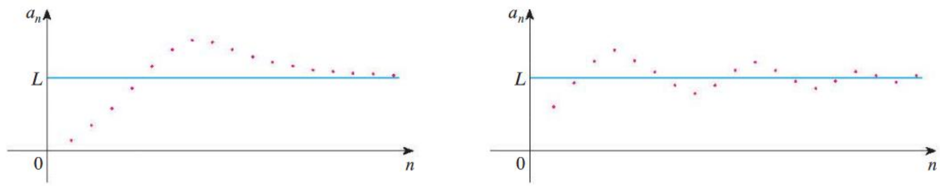
1 Definition A sequence $\{a_n\}$ has the **limit** L and we write

$$\lim_{n\to\infty} a_n = L \qquad \text{or} \qquad a_n \to L \text{ as } n \to \infty$$

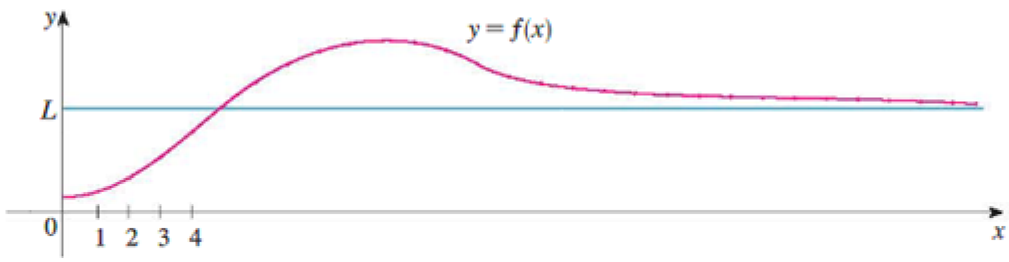
if we can make the terms a_n as close to L as we like by taking n sufficiently large. If $\lim_{n\to\infty} a_n$ exists, we say the sequence **converges** (or is **convergent**). Otherwise, we say the sequence **diverges** (or is **divergent**).

Ex6: Find $\lim_{n\to\infty}(n+1)$.

The following are examples of two sequences that converge to L:



3 Theorem If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$ when n is an integer, then $\lim_{n\to\infty} a_n = L$.



In particular, since we know that $\lim_{x\to\infty} (1/x^r) = 0$ when $r > 0$, we have

4
$$\lim_{n\to\infty} \frac{1}{n^r} = 0 \qquad \text{if } r > 0$$

If a_n becomes large as n becomes large, we use the notation $\lim_{n\to\infty} a_n = \infty$.

If $\lim_{n\to\infty} a_n = \infty$, then the sequence $\{a_n\}$ is divergent but in a special way. We say that $\{a_n\}$ diverges to ∞.

The limit laws given in calculus I also hold for the limits of sequences and have similar proofs.

If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is a constant, then

$$\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} (a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} ca_n = c \lim_{n\to\infty} a_n \qquad\qquad \lim_{n\to\infty} c = c$$

$$\lim_{n\to\infty} (a_n b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} \quad \text{if } \lim_{n\to\infty} b_n \neq 0$$

$$\lim_{n\to\infty} a_n^p = \left[\lim_{n\to\infty} a_n\right]^p \quad \text{if } p > 0 \text{ and } a_n > 0$$

<u>Ex7</u>: Find the following limits.

a) $\lim_{n\to\infty}(-1)^n$

b) $\lim_{n\to\infty}\frac{n+4}{n+1}$

c) $\lim_{n\to\infty}\frac{n+\ln n}{n^2}$

The Squeeze Theorem can also be adapted for sequences as follows (see Figure 7).

If $a_n \leqslant b_n \leqslant c_n$ for $n \geqslant n_0$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.

6 Theorem If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

Proof:

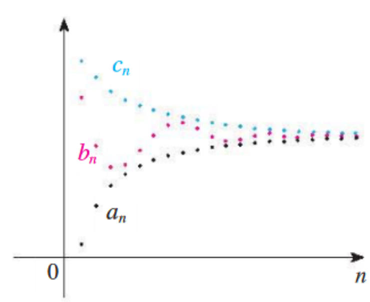


FIGURE 7
The sequence $\{b_n\}$ is squeezed between the sequences $\{a_n\}$ and $\{c_n\}$.

7 Theorem If $\lim_{n\to\infty} a_n = L$ and the function f is continuous at L, then

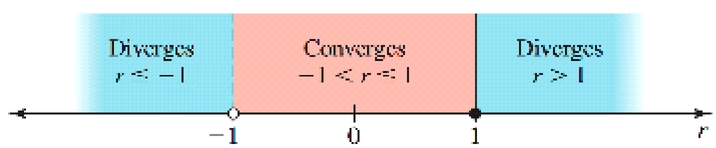
$$\lim_{n\to\infty} f(a_n) = f(L)$$

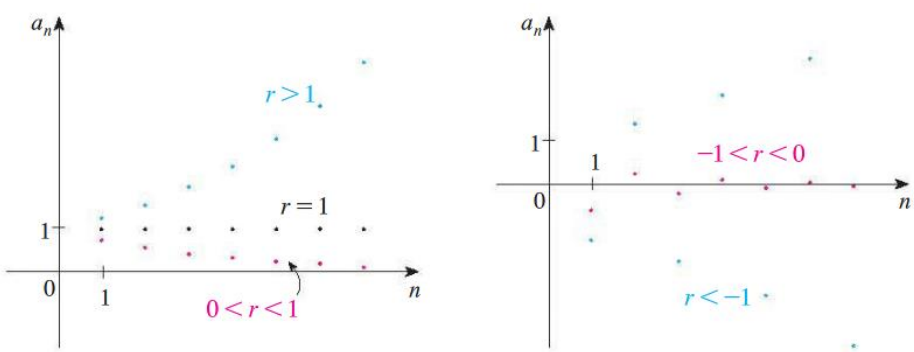
Ex8: Find the following limits.

a) $\lim_{n\to\infty} e^{\frac{3n}{n+1}}$

b) $\lim_{n\to\infty} \sqrt{\frac{n+1}{n}}$

Note: $\lim_{n\to\infty} r^n = \begin{cases} 0 & if \quad -1 < r < 1 \\ 1 & if \quad r = 1 \\ \text{diverges} & if \quad r > 1 \text{ or } r \le -1 \end{cases}$





10 **Definition** A sequence $\{a_n\}$ is called **increasing** if $a_n < a_{n+1}$ for all $n \geqslant 1$, that is, $a_1 < a_2 < a_3 < \cdots$. It is called **decreasing** if $a_n > a_{n+1}$ for all $n \geqslant 1$. A sequence is **monotonic** if it is either increasing or decreasing.

11 **Definition** A sequence $\{a_n\}$ is **bounded above** if there is a number M such that

$$a_n \leqslant M \qquad \text{for all } n \geqslant 1$$

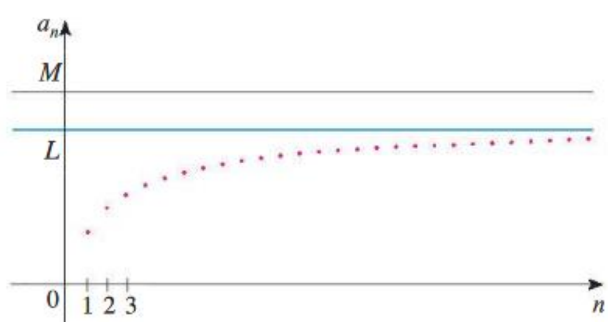
It is **bounded below** if there is a number m such that

$$m \leqslant a_n \qquad \text{for all } n \geqslant 1$$

If it is bounded above and below, then $\{a_n\}$ is a **bounded sequence**.

For instance, the sequence $a_n = n$ is bounded below $(a_n > 0)$ but not above. The sequence $a_n = n/(n+1)$ is bounded because $0 < a_n < 1$ for all *n*.

We know that not every bounded sequence is convergent [for instance, the sequence $a_n = (-1)^n$ satisfies $-1 \le a_n \le 1$ but is divergent from Example 7a and not every monotonic sequence is convergent $(a_n = n \to \infty)$. But if a sequence is both bounded *and* monotonic, then it must be convergent. Intuitively you can understand why by looking at the figure. If $\{a_n\}$ is increasing and $a_n \le M$ for all *n*, then the terms are forced to crowd together and approach some number *L*.



12 **Monotonic Sequence Theorem** Every bounded, monotonic sequence is convergent.

Ex9: Verify that $a_n = \sqrt{n+1} - \sqrt{n}$ is decreasing and bounded below. Does $\lim_{n \to \infty} a_n$ exist?

<u>Ex10</u>: Calculate $\lim_{n\to\infty} a_n$ if $a_1 = \sqrt{2}, \quad a_2 = \sqrt{2\sqrt{2}}, \quad a_3 = \sqrt{2\sqrt{2\sqrt{2}}}, \ldots$