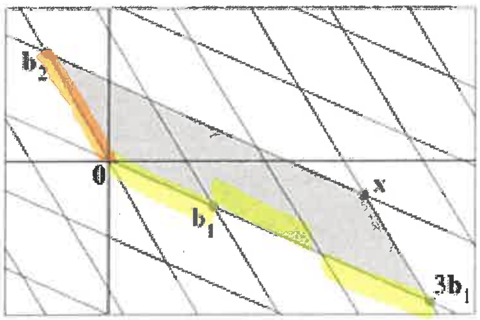
4.7: Change of Basis

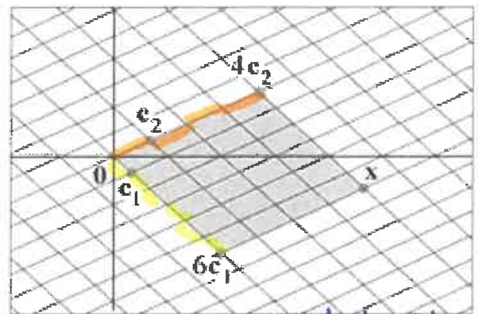
Math 220: Linear Algebra

We are now going to look at converting a vector x in one coordinate system into another coordinate system – same vector, different coordinate representation.

Consider the following vector spaces spanned by $\{\mathbf{b}_1, \mathbf{b}_2\}$ and $\{\mathbf{c}_1, \mathbf{c}_2\}$ respectively.



(a)



(b)

standard grid $\vec{x}$

parallelogram grid $[\vec{x}]_B$

$P_B = [\vec{b}_1 \ \vec{b}_2]$

standard grid $\vec{x}$

parallelogram grid $[\vec{x}]_C$

$P_C = [\vec{c}_1 \ \vec{c}_2]$

By observation, find $[\mathbf{x}]_B = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $[\mathbf{x}]_C = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$

$3\vec{b}_1 + 1\vec{b}_2$ $\qquad$ $6\vec{c}_1 + 4\vec{c}_2$

Ex 1: Consider two bases $B = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $C = \{\mathbf{c}_1, \mathbf{c}_2\}$ for a vector space V, such that

$$\mathbf{b}_1 = 4\mathbf{c}_1 + \mathbf{c}_2 \quad \text{and} \quad \mathbf{b}_2 = -6\mathbf{c}_1 + \mathbf{c}_2$$

Suppose $\mathbf{x} = 3\mathbf{b}_1 + \mathbf{b}_2$ (that is, $[\mathbf{x}]_B = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$), find $[\mathbf{x}]_C$.

$$\Rightarrow [\vec{x}]_C = [3\vec{b}_1 + \vec{b}_2]_C$$

$$= 3[\vec{b}_1]_C + 1[\vec{b}_2]_C$$

$$= \begin{bmatrix} | & | \\ [\vec{b}_1]_C & [\vec{b}_2]_C \\ | & | \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -6 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

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Theorem 15
Let $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ and $C = \{\mathbf{c}_1, \ldots, \mathbf{c}_n\}$ be bases of a vector space V. Then there is a unique $n \times n$ matrix $\underset{C\leftarrow B}{P}$ such that

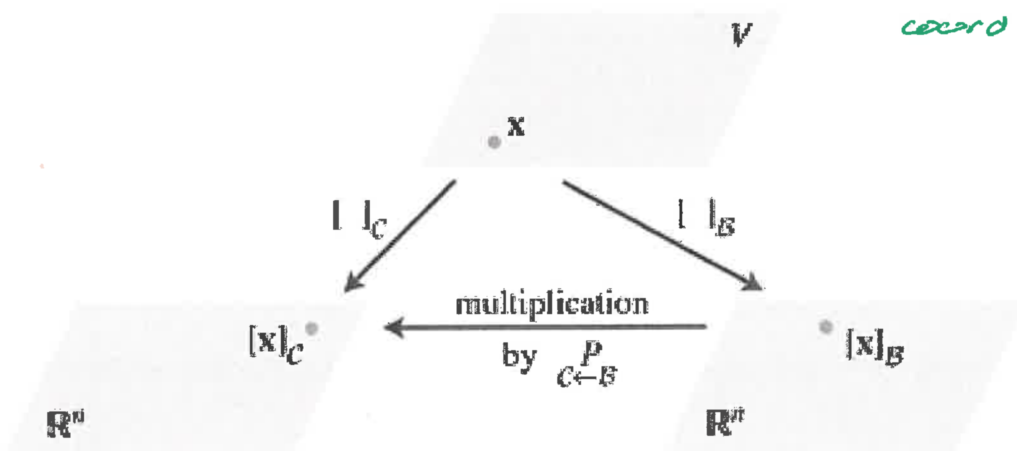
$$[\mathbf{x}]_C = \underset{C\leftarrow B}{P}[\mathbf{x}]_B \qquad (4)$$

The columns of $\underset{C\leftarrow B}{P}$ are the C-coordinate vectors of the vectors in the basis B. That is,

$$\underset{C\leftarrow B}{P} = \left[[\mathbf{b}_1]_C \quad [\mathbf{b}_2]_C \quad \cdots \quad [\mathbf{b}_n]_C \right]$$

Pro tip: To remember this, think of $\underset{C\leftarrow B}{P}$ as a linear combination of the columns of $\underset{C\leftarrow B}{P}$. The matrix product is a C-coord vector, so the cols of $\underset{C\leftarrow B}{P}$ should also be C-coord vectors.

$\underset{C\leftarrow B}{P}$ is the change of coordinates matrix from B to C



Why are the columns of $\underset{C\leftarrow B}{P}$ linearly independent? The columns form a basis for the vector space $\Rightarrow$ Linearly independent.

So $\underset{C\leftarrow B}{P}$ is invertible.

So equation (4) above can be re-written as $\left(\underset{C\leftarrow B}{P}\right)^{-1}[\mathbf{x}]_C = [\mathbf{x}]_B$

Since $\underset{C\leftarrow B}{P}$ is the matrix that converts B-coordinates to C-coordinates, what should $\left(\underset{C\leftarrow B}{P}\right)^{-1}$ do? It should convert from C-coordinates to B-coordinates.

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$$\boxed{\left(\underset{C\leftarrow B}{P}\right)^{-1} = \underset{B\leftarrow C}{P}}$$

Change of Basis in $\mathbb{R}^n$

If $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ and ε is the *standard basis* $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ in $\mathbb{R}^n$, then $[\mathbf{b}_1]_\varepsilon = \mathbf{b}_1$, and likewise for the other vectors in B. In this case, $\underset{\varepsilon\leftarrow B}{P}$ is the same as the change-of-coordinates matrix P_B introduced in Section 4.4, namely,

$$P_B = [\mathbf{b}_1 \quad \mathbf{b}_2 \quad \cdots \quad \mathbf{b}_n]$$

However, to change coordinates between two non-standard bases in $\mathbb{R}^n$, we will need to use Theorem 15 and find coordinate vectors of the old basis relative to the new basis.

Ex 2:

Let $\mathbf{b}_1 = \begin{bmatrix} -6 \\ -1 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \mathbf{c}_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \mathbf{c}_2 = \begin{bmatrix} 6 \\ -2 \end{bmatrix}$ and consider the bases for $\mathbb{R}^2$ given by $B = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $C = \{\mathbf{c}_1, \mathbf{c}_2\}$. Find the change-of-coordinates matrix from B to C.

we want to find $\underset{C\leftarrow B}{P}$

we need $\underbrace{\vec{b}_1 \;\&\; \vec{b}_2}_{\text{old}}$ in terms of $\underbrace{\vec{c}_1 \;\&\; \vec{c}_2}_{\text{new}}$

$\Rightarrow$ solve $\vec{b}_1 = \begin{bmatrix} | & | \\ \vec{c}_1 & \vec{c}_2 \\ | & | \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $\vec{b}_2 = \begin{bmatrix} | & | \\ \vec{c}_1 & \vec{c}_2 \\ | & | \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

$\Rightarrow$ solve $\begin{bmatrix} 2 & 6 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -6 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 2 & 6 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

$\Rightarrow$ row reduce $\left[\begin{array}{cc|c} 2 & 6 & -6 \\ -1 & -2 & -1 \end{array}\right]$ and $\left[\begin{array}{cc|c} 2 & 6 & 2 \\ -1 & -2 & 0 \end{array}\right]$

which combine to $\left[\begin{array}{cc|cc} 2 & 6 & -6 & 2 \\ -1 & -2 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & 9 & -2 \\ 0 & 1 & -4 & 1 \end{array}\right]$

$\Rightarrow$ $\vec{b}_1 = 9\vec{c}_1 - 4\vec{c}_2$ and $\vec{b}_2 = -2\vec{c}_1 + 1\vec{c}_2$

or $[\vec{b}_1]_C = \begin{bmatrix} 9 \\ -4 \end{bmatrix}_C$ or $[\vec{b}_2]_C = \begin{bmatrix} -2 \\ 1 \end{bmatrix}_C$

conclusion

$$\boxed{\underset{C\leftarrow B}{P} = \begin{bmatrix} 9 & -2 \\ -4 & 1 \end{bmatrix}}$$

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The method: In order to find the change of coordinates matrix, row reduce.

$$\boxed{\left[\begin{array}{cc|cc} \mathbf{c}_1 & \mathbf{c}_2 & \mathbf{b}_1 & \mathbf{b}_2 \end{array}\right] \sim \left[\begin{array}{c|c} I & \underset{C\leftarrow B}{P} \end{array}\right]}$$

new   old

Ex 3: Let $\mathbf{b}_1 = \begin{bmatrix} 7 \\ 5 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} -3 \\ -1 \end{bmatrix}, \mathbf{c}_1 = \begin{bmatrix} 1 \\ -5 \end{bmatrix}, \mathbf{c}_2 = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$ and consider the bases for $\mathbb{R}^2$ given by $B = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $C = \{\mathbf{c}_1, \mathbf{c}_2\}$.

a. Find the change-of-coordinates matrix from *C* to *B*.

b. Find the change-of-coordinates matrix from *B* to *C*.

(a) $\left[\begin{array}{cc|cc} 7 & -3 & 1 & -2 \\ 5 & -1 & -5 & 2 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 1 & -5 & 3 \end{array}\right]$

so $\underset{B\leftarrow C}{P} = \begin{bmatrix} -2 & 1 \\ -5 & 3 \end{bmatrix}$

(b) $\left[\begin{array}{cc|cc} 1 & -2 & 7 & -3 \\ -5 & 2 & 5 & -1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & -3 & 1 \\ 0 & 1 & -5 & 2 \end{array}\right]$

so $\underset{C\leftarrow B}{P} = \begin{bmatrix} -3 & 1 \\ -5 & 2 \end{bmatrix}$

Practice Problems

1. Let $F = \{\mathbf{f}_1, \mathbf{f}_2\}$ and $G = \{\mathbf{g}_1, \mathbf{g}_2\}$ be bases for a vector space *V*, and let *P* be a matrix whose columns are $[\mathbf{f}_1]_G$ and $[\mathbf{f}_2]_G$. Which of the following equations is satisfied by *P* for all **v** in *V*?

$P = \begin{bmatrix} [f_1]_G & [f_2]_G \end{bmatrix}$

(i) $[\mathbf{v}]_F = P[\mathbf{v}]_G$

* (ii) $[\mathbf{v}]_G = P[\mathbf{v}]_F$ This is $\underset{G\leftarrow F}{P}$

2. Let *B* and *C* be as in Example 1. Use the results of that example to find the change-of-coordinates matrix from *C* to *B*.

$\underset{C\leftarrow B}{P} = \begin{bmatrix} 4 & -6 \\ 1 & 1 \end{bmatrix} \Rightarrow \underset{B\leftarrow C}{P} = \frac{1}{10}\begin{bmatrix} 1 & 6 \\ -1 & 4 \end{bmatrix}$

check:

$\underbrace{\frac{1}{10}\begin{bmatrix} 1 & 6 \\ -1 & 4 \end{bmatrix}}_{\underset{B\leftarrow C}{P}} \underbrace{\begin{bmatrix} 6 \\ 4 \end{bmatrix}}_{[\vec{x}]_C} = \underbrace{\begin{bmatrix} 3 \\ 1 \end{bmatrix}}_{[\vec{x}]_B}$