4.5 & 4.6: The Dimension of a Vector Space, Rank

Math 220: Linear Algebra

Theorem 9

If a vector space *V* has a basis $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$, then any set in *V* containing more than *n* vectors must be linearly dependent.

Theorem 10

If a vector space *V* has a basis of *n* vectors, then every basis of *V* must consist of exactly *n* vectors.

Definition

If *V* is spanned by a finite set, then *V* is said to be **finite-dimensional,** and the **dimension** of *V*, written as dim *V*, is the number of vectors in a basis for *V*. The dimension of the zero vector space $\{\mathbf{0}\}$ is defined to be zero. If *V* is not spanned by a finite set, then *V* is said to be **infinite-dimensional.**

Ex 1: Find the following

a) $\dim \mathbb{R}^n =$ n

b) $\dim P_3 =$ 4 $\longleftrightarrow$ $\left(P_3 = \text{Span}\{1, t, t^2, t^3\}\right)$

c) $\dim P_n =$ $n+1$

d) $\dim P$. P is infinite Dimensional $\longleftrightarrow$ (P = all polynomials)

e) $\dim H =$ 2 Given $H = \text{span}\left\{\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}4\\5\\6\end{bmatrix}\right\}$

f) $\dim G =$ 1 Given $G = \text{span}\left\{\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}2\\4\\6\end{bmatrix}\right\}$

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Ex 2: Find the dimension of the subspace

$$H = \left\{ \begin{bmatrix} a - 4b - 2c \\ 2a + 5b - 4c \\ -a + 2c \\ -3a + 7b + 6c \end{bmatrix} : a, b, c \text{ in } \mathbb{R} \right\}$$

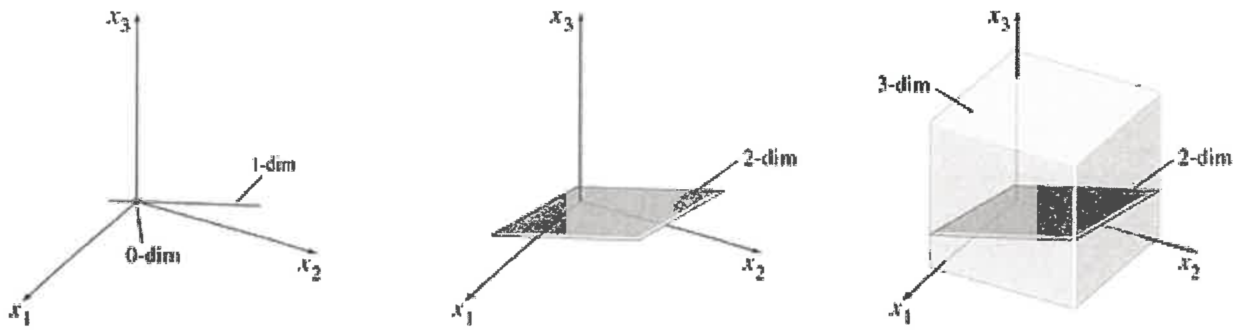
write H as:

$$a\begin{bmatrix}1\\2\\-1\\-3\end{bmatrix} + b\begin{bmatrix}-4\\5\\0\\7\end{bmatrix} + c\begin{bmatrix}-2\\-4\\2\\6\end{bmatrix}$$

$$\Rightarrow \text{rref}\Big(\underbrace{\begin{bmatrix}1 & -4 & -2\\2 & 5 & -4\\-1 & 0 & 2\\-3 & 7 & 6\end{bmatrix}}_{A}\Big) \sim \begin{bmatrix}1 & 0 & -2\\0 & 1 & 0\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix} \Rightarrow \text{basis for col } A \ \left\{\begin{bmatrix}1\\2\\-1\\-3\end{bmatrix}, \begin{bmatrix}-4\\5\\0\\7\end{bmatrix}\right\}$$

so $\dim H = 2$.

The subspaces of $\mathbb{R}^3$ can be classified by dimension now.



Theorem 11
Let *H* be a subspace of a finite-dimensional vector space *V*. Any linearly independent set in *H* can be expanded, if necessary, to a basis for *H*. Also, *H* is finite-dimensional and

$$\dim H \leq \dim V$$

Proof:

case 1: If $H = \{\vec{0}\}$ then $\dim H = 0 \leq \dim V$.

case 2: Let H be a subspace that includes L.I. vectors $S = \{\vec{v}_1, \ldots, \vec{v}_k\}$. If S is not a basis for H, then there is a $\vec{v}_{k+1} \in H$ s.t. $\vec{v}_{k+1} \notin \text{span } S$.

$\Rightarrow \{\vec{v}_1, \ldots, \vec{v}_k, \vec{v}_{k+1}\}$ is LI (by Thm 4 in section 4.3)

As long as the new set doesn't span H, repeat.

Eventually a new S will span H and be a basis.

Also $\dim H \leq \dim V$ as a corollary to Thm 9.

Q.E.D.

Theorem 12 The Basis Theorem
Let *V* be a *p*-dimensional vector space, $p \geq 1$. Any linearly independent set of exactly *p* elements in *V* is automatically a basis for *V*. Any set of exactly *p* elements that spans *V* is automatically a basis for *V*.

basis L.I. dim
span

Proof:

(1) By Thm 11, any linearly independent set S of p elements can be expandeded to span V. But dim V = p so the basis must contain exactly p L.I. vectors.
$\Rightarrow$ S is a basis.

(2) S has p elements that span V. The spanning set Thm (section 4.3) implies there is a subset $S^* \subset S$ that is a basis for V. But dim V = p so S^* must contain p vectors. Thus $S^* = S \Rightarrow S$ is a basis.
QED.

An example: What can we say about the dimension of Col A and Nul A?

$$A = \begin{bmatrix} a & b & c & d & e \\ f & g & h & i & j \\ k & l & m & n & o \end{bmatrix} \sim \begin{bmatrix} 1 & r & s & 0 & 0 \\ 0 & 0 & 0 & 1 & t \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

dim(col A) = 2
and
dim(Nul A) = 3

The dimension of the null space of A is

The number of free variables in rref(A)

The dimension of the column space of A is:

The number of pivots in A.

Ex 3: Determine the dimensions of the null space and the column space of A.

$$A = \begin{bmatrix} 1 & 0 & -3 & 1 & 2 \\ 0 & 1 & -4 & -3 & 1 \\ -3 & 2 & 1 & -8 & -6 \\ 2 & -3 & 6 & 7 & 9 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & -3 & 0 & 4 \\ 0 & 1 & -4 & 0 & -5 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

dim(col A) = 3 (rank)

and

dim(Nul A) = 2 (nullity)

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Row Space

The set of all the linear combinations of the row vectors of an $m \times n$ matrix *A* is called the row space of *A*, and is denoted by Row A. Since there are *n* entries in each row, Row *A* is a subspace of $\mathbb{R}^n$. Also, Row *A* = col A^T.

Ex 4: Find a spanning set for Row *A*.

$$A = \begin{bmatrix} 1 & 0 & -3 & 1 & 2 \\ 0 & 1 & -4 & -3 & 1 \\ -3 & 2 & 1 & -8 & -6 \\ 2 & -3 & 6 & 7 & 9 \end{bmatrix}$$

$\vec{r}_1 = (1, 0, -3, 1, 2)$
$\vec{r}_2 = (0, 1, -4, -3, 1)$
$\vec{r}_3 = (-3, 2, 1, -8, -6)$
$\vec{r}_4 = (2, -3, 6, 7, 9)$

and Row A = span$\{\vec{r}_1, \vec{r}_2, \vec{r}_3, \vec{r}_4\}$

Theorem 13
If two matrices *A* and *B* are row equivalent, then their row spaces are the same. If *B* is in echelon form, the nonzero rows of *B* form a basis for the row space of *A* as well as for that of *B*.

Ex 5: Find bases for the row space, column space, and null space of A.

$$A = \begin{bmatrix} 1 & 0 & -3 & 1 & 2 \\ 0 & 1 & -4 & -3 & 1 \\ -3 & 2 & 1 & -8 & -6 \\ 2 & -3 & 6 & 7 & 9 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & -3 & 0 & 4 \\ 0 & 1 & -4 & 0 & -5 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

row space has basis: $\{(1,0,-3,0,4), (0,1,-4,0,-5), (0,0,0,1,-2)\}$

col A has basis: $\left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ -8 \\ 7 \end{bmatrix} \right\}$

Nul A has basis: $\left\{ \begin{bmatrix} 3 \\ 4 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 5 \\ 0 \\ 2 \\ 1 \end{bmatrix} \right\}$

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The rank of A is the dimension of the column space of A.

The rank of A^T is the dimension of the row space of A.

The nullity of A is the dimension of the null space of A (though this text just uses dim(Nul A).)

Theorem 14 The Rank Theorem
The dimensions of the column space and the row space of an $m \times n$ matrix A are equal. This common dimension, the rank of A, also equals the number of pivot positions in A and satisfies the equation

$$\text{rank } A + \dim \text{Nul } A = n$$

(See proof on page 235.)

Ex 6: a) If A is an 42 x 35 matrix with three-dimensional null space, what is the rank of A?

rank + 3 = 35 $\Rightarrow$ rank = 32

b) Could a 3x5 matrix have a one-dimensional null space?

max rank = 3

rank + nullity = columns in A

3 + 1 $\neq$ 5 so, "no"

In chapter 6 we will learn that Row A and Nul A have only the zero vector in common, and they are actually perpendicular to each other. ***Take a look at example 4 on page 236.***

Ex 7: A scientist has found two solutions to a homogeneous system of 40 equations in 42 variables. The two solutions are not multiples, and all other solutions can be constructed by adding together appropriate multiples of these two solutions. Can the scientist be *certain* that an associated nonhomogeneous system (with the same coefficients) has a solution?

$\Rightarrow$ nullity ≥ 2 and there are no other L.I. soln

so nullity = 2,

$\Rightarrow$ 42 − 2 = 40 = rank

$\therefore$ "yes" All vectors in $\mathbb{R}^{40}$ are in Col A.

Theorem The Invertible Matrix Theorem (continued)
Let A be an $n \times n$ matrix. Then the following statements are each equivalent to the statement that A is an invertible matrix.

m. The columns of A form a basis of $\mathbb{R}^n$.

n. $\text{Col } A = \mathbb{R}^n$

o. $\dim \text{Col } A = n$

p. $\text{rank } A = n$

q. $\text{Nul } A = \{\mathbf{0}\}$

r. $\dim \text{Nul } A = 0$

Practice Problems
The matrices below are row equivalent.

$$A = \begin{bmatrix} 2 & -1 & 1 & -6 & 8 \\ 1 & -2 & -4 & 3 & -2 \\ -7 & 8 & 10 & 3 & -10 \\ 4 & -5 & -7 & 0 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -2 & -4 & 3 & -2 \\ 0 & 3 & 9 & -12 & 12 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

1. Find rank A and dim Nul A. rank $A = 2$ Nullity $= 5 - 2 = 3$

2. Find bases for Col A and Row A.

3. What is the next step to perform to find a basis for Nul A?

4. How many pivot columns (rank) are in a row echelon form of A^T?

(2) basis for col A: $\left\{ \begin{bmatrix} 2 \\ 1 \\ -7 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 8 \\ -5 \end{bmatrix} \right\}$

basis for row A: $\{(1,-2,-4,3,-2), (0,3,9,-12,12)\}$

(3) Finish row reducing B. $\frac{1}{3}R_2 \to R_2$

(4) $2 \leftarrow$ rank of $A^T = \dim(\text{row } A)$.