4.4: Coordinate Systems

Math 220: Linear Algebra

Theorem 7 The Unique Representation Theorem
Let $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis for a vector space *V*. Then for each **x** in *V*, there exists a unique set of scalars $c_1, \ldots, c_n$ such that

WRT: with respect to

$$\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n$$

Proof:

Let $\vec{x} \in V$ be given and suppose there are two representations for x WRT B. $\vec{x} = c_1\vec{b}_1 + \ldots + c_n\vec{b}_n$ and $\vec{x} = d_1\vec{b}_1 + \ldots + d_n\vec{b}_n$.

$\Rightarrow \vec{0} = \vec{x} - \vec{x} = (c_1 - d_1)\vec{b}_1 + \ldots + (c_n - d_n)\vec{b}_n$

since B is a basis and linearly independent, there is only the trivial solution to the homogeneous equation

$\Rightarrow c_i - d_i = 0$ for $i = 1, 2, \ldots, n$ $\Rightarrow c_i = d_i$

$\therefore$ $\vec{x}$ has a unique representation.

Definition
Suppose $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ is a basis for *V* and **x** is in *V*. The **coordinates of x relative to the basis *B*** (or the ***B*-coordinates of x**) are the weights $c_1, \ldots, c_n$ such that $\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n$.

We call this vector the coordinate vector of $\vec{x}$ (relative to basis B) or the B-coordinate vector of $\vec{x}$

$$[\mathbf{x}]_B = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

$\mathbf{x} \mapsto [\mathbf{x}]_B$ is the coordinate mapping (determined by *B*)

Ex 1: Consider a basis $B = \{\mathbf{b}_1, \mathbf{b}_2\}$ for $\mathbb{R}^2$, where $\mathbf{b}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{b}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Suppose an **x** in $\mathbb{R}^2$ has the coordinate vector $[\mathbf{x}]_B = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$. Find **x**.

coordinates WRT $\vec{b}_1$ & $\vec{b}_2$

coordinates WRT $\vec{e}_1$ and $\vec{e}_2$

$\vec{x} = -2\vec{b}_1 + 3\vec{b}_2$

$= -2\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$= \begin{bmatrix} 1 \\ 6 \end{bmatrix}$

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Ex 2: The entries in the vector $\mathbf{x} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$ are the coordinates of **x** relative to the *standard basis* $\varepsilon = \{\mathbf{e}_1, \mathbf{e}_2\}$, since

$$\begin{bmatrix} 1 \\ 6 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 6\begin{bmatrix} 0 \\ 1 \end{bmatrix} = 1\vec{e}_1 + 6\vec{e}_2$$

If $\varepsilon = \{\mathbf{e}_1, \mathbf{e}_2\}$, then $[\mathbf{x}]_\varepsilon = \mathbf{x}$.

$[\vec{x}]_B = -2\vec{b}_1 + 3\vec{b}_2 = \begin{bmatrix} -2 \\ 3 \end{bmatrix}_B$

$[x]_\varepsilon = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$

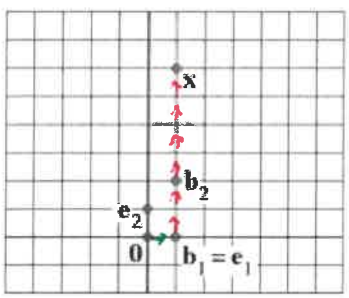
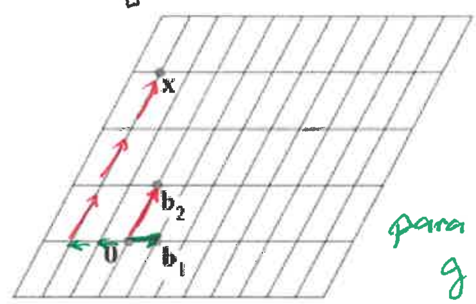


FIGURE 1 Standard graph paper.



parallelogram grid

FIGURE 2 *B*-graph paper.

See Example 3 on page 219.

Ex 3: Let $\mathbf{b}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$, and $B = \{\mathbf{b}_1, \mathbf{b}_2\}$. Find the coordinate vector $[\mathbf{x}]_B$ of **x** relative to *B*.

using row reduction

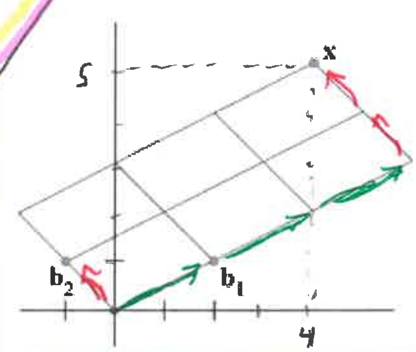
$$c_1\begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix} \Rightarrow \underbrace{\begin{array}{cc} \vec{b}_1 & \vec{b}_2 \\ \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \end{array}}_{P_B} \underbrace{\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}}_{[\vec{x}]_B} = \begin{array}{c} \vec{x} \\ \begin{bmatrix} 4 \\ 5 \end{bmatrix} \end{array}$$

$$\Rightarrow \text{rref}\left(\begin{bmatrix} 2 & -1 & | & 4 \\ 1 & 1 & | & 5 \end{bmatrix}\right) \sim \begin{bmatrix} 1 & 0 & | & 3 \\ 0 & 1 & | & 2 \end{bmatrix}$$

Thus $[\vec{x}]_B = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

Graphically

$[\vec{x}]_B = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$



using the change of basis matrix P_B^{-1}

$P_B[\vec{x}]_B = \vec{x} \Rightarrow [\vec{x}]_B = P_B^{-1}\vec{x}$

$$\Rightarrow \begin{bmatrix} 1/3 & 1/3 \\ -1/3 & 2/3 \end{bmatrix}\begin{bmatrix} 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

change of coordinates

The matrix P_B changes the B-coordinates of a vector $\mathbf{x}$ into the standard coordinates for $\mathbf{x}$. An analogous change of coordinates can be carried out for a basis $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$. Let

$$\boxed{P_B = [\mathbf{b}_1 \quad \mathbf{b}_2 \quad \cdots \quad \mathbf{b}_n]}$$

Then the vector equation

$$\mathbf{x} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \cdots + c_n\mathbf{b}_n$$

is equivalent to

$$\boxed{\mathbf{x} = P_B[\mathbf{x}]_B}$$

recall

$[\vec{x}]_B =$

We call P_B the change-of-coordinates matrix from B to the standard bas
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Since the columns of P_B form a basis, they are linearly independent, and hav
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old coordinate

$$P_B^{-1}\mathbf{x} = [\mathbf{x}]_B$$

$\vec{x}$

P_B^{-1}

$[\vec{x}]_B$

new coordinat

4.4: Coordinate Systems

Math 220: Linear Algebra

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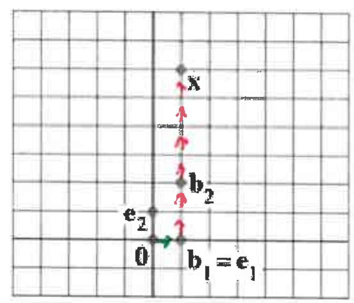
$$\begin{bmatrix} 1 \\ 6 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 6\begin{bmatrix} 0 \\ 1 \end{bmatrix} = 1\vec{e}_1 + 6\vec{e}_2$$

If $\varepsilon = \{\mathbf{e}_1, \mathbf{e}_2\}$, then $[\mathbf{x}]_\varepsilon = \mathbf{x}$.

$$[\vec{x}]_B = -2\vec{b}_1 + 3\vec{b}_2 = \begin{bmatrix} -2 \\ 3 \end{bmatrix}_B$$

$$[x]_\varepsilon = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$$



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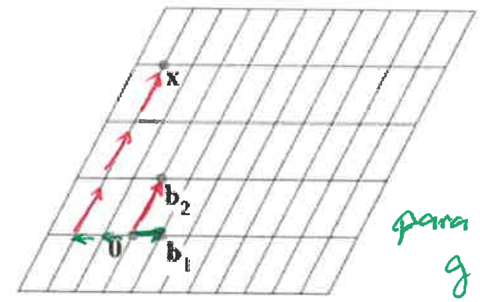

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FIGURE 2 *B*-graph paper.

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Graphically

$$[\vec{x}]_B = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

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The matrix P_B. changes the B-coordinates of a vector **x** into the standard coordinates for **x**. An analogous change of coordinates can be carried out in $\mathbb{R}^n$ for a basis $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$. Let

$$\boxed{P_B = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_n \end{bmatrix}}$$

Then the vector equation

$$\mathbf{x} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \cdots + c_n\mathbf{b}_n$$

$$[\vec{x}]_B = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

is equivalent to

$$\boxed{\mathbf{x} = P_B[\mathbf{x}]_B}$$

recall

We call P_B the change-of-coordinates matrix from B to the standard basis in $\mathbb{R}^n$. Left-multiplication by P_B transforms the coordinate vector $[\mathbf{x}]_B$ into **x**.

Since the columns of P_B form a basis, they are linearly independent, and have an inverse, which leads to

$$P_B^{-1}\mathbf{x} = [\mathbf{x}]_B$$

old coordinate system

$$\begin{tikzcd} \vec{x} \arrow[d, bend right, "P_B^{-1}"'] \\ {[\vec{x}]_B} \arrow[u, bend right, "P_B"'] \end{tikzcd}$$

new coordinate system

4.4: Coordinate Systems

The Coordinate Mapping

Choosing a basis $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ for a vector space *V* introduces a coordinate system in *V*. The coordinate mapping $\mathbf{x} \mapsto [\mathbf{x}]_B$ connects the possibly unfamiliar space *V* to the familiar space $\mathbb{R}^n$. See Figure 5. Points in *V* can now be identified by their new "names."

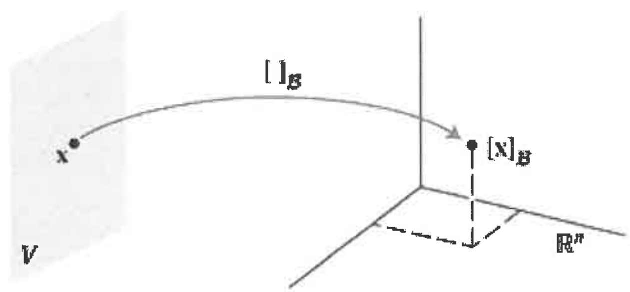


FIGURE 5 The coordinate mapping from *V* onto $\mathbb{R}^n$.

Theorem 8
Let $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis for a vector space *V*. Then the coordinate mapping $\mathbf{x} \mapsto [\mathbf{x}]_B$ is a one-to-one linear transformation from *V* onto $\mathbb{R}^n$.

proof is in the text

A one-to-one linear transformation from a vector space *V* onto a vector space *W* is called an isomorphism from *V* onto *W*.

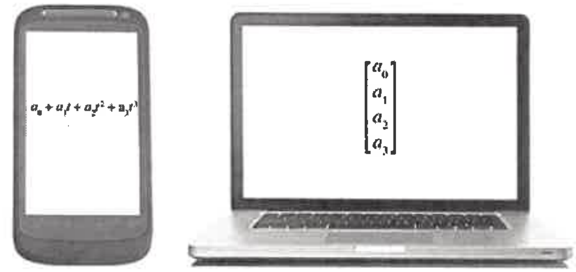
Essentially, these two vector spaces are indistinguishable.

Ex 4: Let *B* be the standard basis of the space $\mathbb{P}_3$ of polynomials; that is, let $B = \{1, t, t^2, t^3\}$. A typical element **p** of $\mathbb{P}_3$ has the form

$$\mathbf{p}(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 \qquad \begin{bmatrix} a_0 \end{bmatrix}$$

So $\mathbf{p} \mapsto [\mathbf{p}]_B$ is an isomorphism

from $\mathbb{P}_3$ onto $\mathbb{R}^4$.



Ex 5: Use coordinate vectors to test the linear independence of the sets of polynomials.

verify L.I. by showing there is only the trivial solution to the homogeneous equation $P_B\vec{x} = \vec{0}$

$\mathbb{P}_3 \leftrightarrow \mathbb{R}^4$

a) $\underset{p_1}{1+2t^3}, \underset{p_2}{2+t-3t^2}, \underset{p_3}{-t+2t^2-t^3}$

$$\underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix}}_{[p_1]_B} \quad \underbrace{\begin{bmatrix} 2 \\ 1 \\ -3 \\ 0 \end{bmatrix}}_{[p_2]_B} \quad \underbrace{\begin{bmatrix} 0 \\ -1 \\ 2 \\ -1 \end{bmatrix}}_{[p_3]_B}$$

$\leftarrow$ use as columns of A.

To verify p_1, p_2, p_3 are L.I. we solve $A\vec{x} = \vec{0}$:

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -3 & 2 & 0 \\ 2 & 0 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

No free variables.

Is this a basis for $\mathbb{P}_3$?

since the vectors are L.I. the polynomials are also L.I.

$\mathbb{P}_3 \leftrightarrow \mathbb{R}^4$

b) $(1-t)^2, t-2t^2+t^3, (1-t)^3$

$$\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \\ -2 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 1 \\ -3 \\ 3 \\ -1 \end{bmatrix}$$

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Ex 6: Let $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} 3 \\ 12 \\ 7 \end{bmatrix}$,

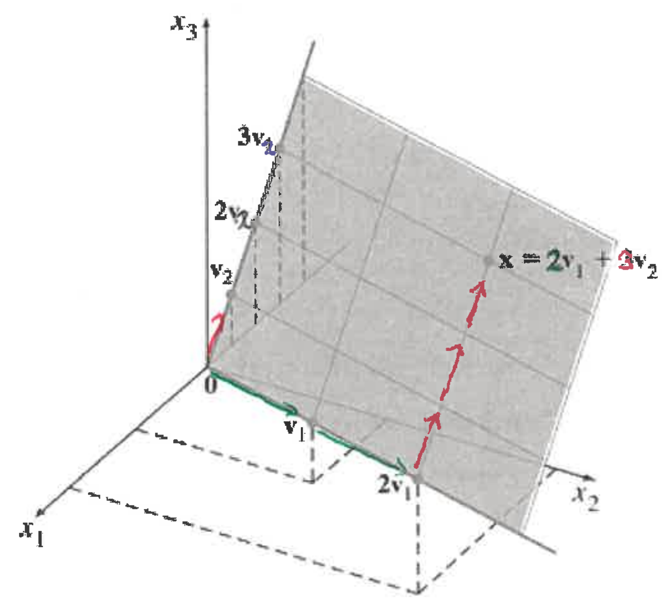
and $B = \{\mathbf{v}_1, \mathbf{v}_2\}$. Then B is a basis for $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$. Determine if **x** is in H, and if it is, find the coordinate vector of **x** relative to B.

we will find $[\vec{x}]_B$ by row reducing.

$$\left[\begin{array}{cc|c} 3 & -1 & 3 \\ 6 & 0 & 12 \\ 2 & 1 & 7 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right] \longleftarrow$$ consistent and thus $\vec{x} \in H$.

$$\Rightarrow \vec{x} = 2\begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix} + 3\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow [\vec{x}]_B = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$



Notice that matrix

$$P_B = \begin{bmatrix} 3 & -1 \\ 6 & 0 \\ 2 & 1 \end{bmatrix}$$ is not

square and so not

4.4: Coordinate Systems

Practice Problems

1. Let $\mathbf{b}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} -3 \\ 4 \\ 0 \end{bmatrix}, \mathbf{b}_3 = \begin{bmatrix} 3 \\ -6 \\ 3 \end{bmatrix}$, and $\mathbf{x} = \begin{bmatrix} -8 \\ 2 \\ 3 \end{bmatrix}$.

a. Show that the set $B = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ is a basis of $\mathbb{R}^3$. Yes, three pivots

b. Find the change-of-coordinates matrix from B to the standard basis.

c. Write the equation that relates **x** in $\mathbb{R}^3$ to $[\mathbf{x}]_B$.

d. Find $[\mathbf{x}]_B$, for the **x** given above.

(b). $\vec{x}$ ... $[\vec{x}]_B$ $\quad P_B = \begin{bmatrix} 1 & -3 & 3 \\ 0 & 4 & -6 \\ 0 & 0 & 3 \end{bmatrix}$

(c) $P_B [\vec{x}]_B = \vec{x}$

(d) $[\vec{x}]_B = P_B^{-1} \vec{x} = \begin{bmatrix} 1 & -3 & 3 \\ 0 & 4 & -6 \\ 0 & 0 & 3 \end{bmatrix}^{-1} \begin{bmatrix} -8 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -5 \\ 2 \\ 1 \end{bmatrix}$ so $[\vec{x}]_B = \begin{bmatrix} -5 \\ 2 \\ 1 \end{bmatrix}$

Alt. solution: $\left[\begin{array}{ccc|c} 1 & -3 & 3 & -8 \\ 0 & 4 & -6 & 2 \\ 0 & 0 & 3 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right]$

2. The set $B = \{1+t, 1+t^2, t+t^2\}$ is a basis for $\mathbb{P}_2$. Find the coordinate vector of $\mathbf{p}(t) = 6 + 3t - t^2$ relative to B.

$\text{rref}\left(\left[\begin{array}{ccc|c} 1 & 1 & 0 & 6 \\ 1 & 0 & 1 & 3 \\ \end{array}\right]\right) \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 1 \\ \end{array}\right]$