2.2: The Inverse of a Matrix

Math 220: Linear Algebra

Remember that the multiplicative inverse or reciprocal of a number, say 7 is $1/7$ or 7^{-1}. The actual definition of this is that

$$\frac{1}{7}\cdot 7 = 1 \qquad 7\cdot\frac{1}{7} = 1$$

An $(n\times n)$ matrix A is called invertible if there is a matrix C such that

$$CA = I \qquad \text{and} \qquad AC = I$$

($I = I_n$ is the $n\times n$ identity matrix.)

Here, C is called the inverse of A. Is C unique?

proof.

Suppose there are two inverses B and C, of matrix A,

$\Rightarrow B = BI = B(AC) = (BA)C = IC = C$

$\Rightarrow B = C$ Q.E.D.

Yes, so denote the inverse with A^{-1} and

$$\boldsymbol{A^{-1}A = I} \quad \textbf{and} \quad \boldsymbol{AA^{-1} = I}$$

A matrix that is **NOT** invertible is called a singular matrix while a matrix that **IS** invertible is called a non-singular matrix.

Ex 1: If $A = \begin{bmatrix} -2 & -3 \\ 3 & 5 \end{bmatrix}$ and $C = \begin{bmatrix} -5 & -3 \\ 3 & 2 \end{bmatrix}$, verify that $C = A^{-1}$.

$$AC = \begin{bmatrix} -2 & -3 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} -5 & -3 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$CA = \begin{bmatrix} -5 & -3 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} -2 & -3 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\therefore C = A^{-1}$

2.2: The Inverse of a Matrix

Theorem 4

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

If $ad - bc = 0$, then A is not invertible.

This value $ad - bc$ is called the determinant and we write

$$\det A = ad - bc$$

So theorem 4 states that A^{-1} exists iff $\det A \neq 0$

Ex 2: Find the inverse of $A = \begin{bmatrix} 3 & -7 \\ 2 & 5 \end{bmatrix}$.

$\det A = 29$

$$A^{-1} = \frac{1}{29}\begin{bmatrix} 5 & 7 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 5/29 & 7/29 \\ -2/29 & 3/29 \end{bmatrix}$$

Theorem 5

If A is an invertible $n \times n$ matrix, then for each $\mathbf{b}$ in $\mathbb{R}^n$, the equation $A\mathbf{x} = \mathbf{b}$ has the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$.

existence and uniqueness proof.

Proof: existence.

Let $\vec{b} \in \mathbb{R}^n$ and invertible $A_{n\times n}$ be given.

solve $A\vec{x} = \vec{b} \Leftrightarrow A^{-1}A\vec{x} = A^{-1}\vec{b} \Leftrightarrow I\vec{x} = A^{-1}\vec{b}$

$\therefore A\vec{x} = \vec{b}$ has a solution $\vec{x} = A^{-1}\vec{b}$.

uniqueness

Let $\vec{u}, \vec{v} \in \mathbb{R}^n$ be solutions to $A\vec{x} = \vec{b}$.

$\Rightarrow A\vec{u} = A\vec{v} = \vec{b} \Rightarrow A^{-1}A\vec{u} = A^{-1}A\vec{v} = A^{-1}\vec{b}$

$\Rightarrow I\vec{u} = I\vec{v} = A^{-1}\vec{b} \Rightarrow \vec{u} = \vec{v}$.

Q.E.D.

2.2: The Inverse of a Matrix

Ex 3: Use the inverse of the matrix $A=\begin{bmatrix}-2 & -3\\ 3 & 5\end{bmatrix}$ from Ex 1 $\left(A^{-1}=\begin{bmatrix}-5 & -3\\ 3 & 2\end{bmatrix}\right)$

to solve the system $\begin{aligned}-2x_1-3x_2&=5\\ 3x_1+5x_2&=-7\end{aligned}$

$A^{-1}\vec{b} = \begin{bmatrix}-5 & -3\\ 3 & 2\end{bmatrix}\begin{bmatrix}5\\ -7\end{bmatrix} = \begin{bmatrix}-4\\ 1\end{bmatrix}$ so $\vec{x} = \begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}-4\\ 1\end{bmatrix}$

Theorem 6

a. If *A* is an invertible matrix, then A^{-1} is invertible and

$$(A^{-1})^{-1} = A$$

b. If *A* and *B* are $n \times n$ invertible matrices, then so is *AB*, and the inverse of *AB* is the product of the inverses of *A* and *B* in the reverse order. That is,

$$(AB)^{-1} = B^{-1}A^{-1}$$

This parallels $(AB)^T = B^T A^T$

c. If *A* is an invertible matrix, then so is A^T, and the inverse of A^T is the transpose of A^{-1}. That is,

$$(A^T)^{-1} = (A^{-1})^T$$

Proofs:

(a) Find C if

$CA^{-1} = I$ and $A^{-1}C = I$.

$\Rightarrow CA^{-1}A = IA$ and $AA^{-1}C = AI$

$\Rightarrow C = A$ and $C = I$

$\therefore$ $C = A$ is the inverse of A^{-1}, or $(A^{-1})^{-1} = A$.

(b) $AB(B^{-1}A^{-1}) = ABB^{-1}A^{-1}$

$= AA^{-1}$

$= I$

AND $(B^{-1}A^{-1})AB = B^{-1}A^{-1}AB$

$= B^{-1}B$

$= I$

$\therefore$ $AB(B^{-1}A^{-1}) = (B^{-1}A^{-1})AB = I$

and $B^{-1}A^{-1} = (AB)^{-1}$.

2.2: The Inverse of a Matrix

(c.) $A^T(A^{-1})^T = (A^{-1}A)^T = I^T = I$

AND $(A^{-1})^T A^T = (AA^{-1})^T = I^T = I$

$\therefore (A^T)^{-1} = (A^{-1})^T$

From Theorem 6b, we can extrapolate to the following.

$(AB)^{-1} = B^{-1}A^{-1}$

The product of $n \times n$ invertible matrices is invertible, and the inverse is the product of their inverses in the reverse order. $(ABCD)^{-1} = D^{-1}C^{-1}B^{-1}A^{-1}$

(Read pages 108-109 on Elementary Matrices)

We are going to look at finding the inverse of a matrix with a slightly different approach than this text.

If an $n \times n$ matrix *A* has an inverse, let's call that matrix *B*. Then

$$AB = I$$

This can be written as $\left[A\vec{b}_1 \; A\vec{b}_2 \cdots A\vec{b}_n \right] = \left[\vec{e}_1 \; \vec{e}_2 \cdots \vec{e}_n \right]$

We can think of this as many systems, where each solution forms the columns vectors of our matrix B.

$A\vec{b}_1 = \vec{e}_1$

$A\vec{b}_2 = \vec{e}_2 \Rightarrow \left[A \,\vdots\, \vec{e}_1 \; \vec{e}_2 \cdots \vec{e}_n \right]$

$\vdots$

$A\vec{b}_n = \vec{e}_n \qquad = [A \,\vdots\, I]$

We could solve each one of these individually, or stack them all together.

Ex 4: Find the inverse of $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 4 \\ 1 & -1 & 10 \end{bmatrix}$.

$[A \,\vdots\, I]$

$$= \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 4 & 0 & 1 & 0 \\ 1 & -1 & 10 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 54 & -23 & -7 \\ 0 & 1 & 0 & -16 & 7 & 2 \\ 0 & 0 & 1 & -7 & 3 & 1 \end{array}\right] = [I \,\vdots\, A^{-1}]$$

so $A^{-1} = \begin{bmatrix} 54 & -23 & -7 \\ -16 & 7 & 2 \\ -7 & 3 & 1 \end{bmatrix}$

Theorem 7
An $n \times n$ matrix *A* is invertible if and only if *A* is row equivalent to I_n, and in this case, any sequence of elementary row operations that reduces *A* to I_n also transforms I_n into A^{-1}.

$$[A \,\vdots\, I] \sim [I \,\vdots\, A^{-1}]$$

Algorithm for Finding A^{-1}
Row reduce the augmented matrix $[A \quad I]$. If *A* is row equivalent to *I*, then $[A \quad I]$ is row equivalent to $[I \quad A^{-1}]$. Otherwise, *A* does not have an inverse.

Ex 5: Find the inverse of the matrix $A = \begin{bmatrix} 1 & -2 & -1 \\ -1 & 5 & 6 \\ 5 & -4 & 5 \end{bmatrix}$, if it exists.

(Do this by hand – more practice.)

$[A \quad I]$

$$= \begin{bmatrix} 1 & -2 & -1 & 1 & 0 & 0 \\ -1 & 5 & 6 & 0 & 1 & 0 \\ 5 & -4 & 5 & 0 & 0 & 1 \end{bmatrix}$$

$R_2 + R_1 \to R_2$ AND $R_3 - 5R_1 \to R_3$

$$\sim \begin{bmatrix} 1 & -2 & -1 & 1 & 0 & 0 \\ 0 & 3 & 5 & 1 & 1 & 0 \\ 0 & 6 & 10 & -5 & 0 & 1 \end{bmatrix}$$

$R_3 - 2R_2 \to R_3$

$$\sim \begin{bmatrix} 1 & -2 & -1 & 1 & 0 & 0 \\ 0 & 3 & 5 & 1 & 1 & 0 \\ 0 & 0 & 0 & -7 & -2 & 1 \end{bmatrix}$$

A is singular. (not invertible)