1.5: Solution Sets of Linear Systems

Math 220: Linear Algebra

A system of linear equations is called homogeneous if it can be written as $A\mathbf{x}=0$ Such a system always has the trivial solution $\vec{0}$.

The important question is whether or not there is a non-trivial solution to a homogeneous system.

Since there is always a trivial solution, there is ~~only~~ a non-trivial solution if and only if there is at least one free variable.

Ex 1: Determine whether the following has a non-trivial solution, and if so, describe the solution set.

$$\begin{aligned} 2x_1 - 5x_2 + 8x_3 &= 0 \\ -2x_1 - 7x_2 + x_3 &= 0 \\ 4x_1 + 2x_2 + 7x_3 &= 0 \end{aligned} \Rightarrow \begin{bmatrix} 2 & -5 & 8 & 0 \\ -2 & -7 & 1 & 0 \\ 4 & 2 & 7 & 0 \end{bmatrix}$$

$$\sim \begin{array}{c} \begin{bmatrix} 1 & 0 & \frac{17}{8} & 0 \\ 0 & 1 & -3/4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ \uparrow \end{array} \qquad \begin{aligned} x_1 &= -\tfrac{17}{8}x_3 \\ x_2 &= \tfrac{3}{4}x_3 \\ x_3 &= x_3 \text{ (free} \end{aligned}$$

free variable and so non-trivial solution

The solutions are vectors of the form $\vec{x} = x_3 \begin{bmatrix} -17/8 \\ 3/4 \\ 1 \end{bmatrix}$

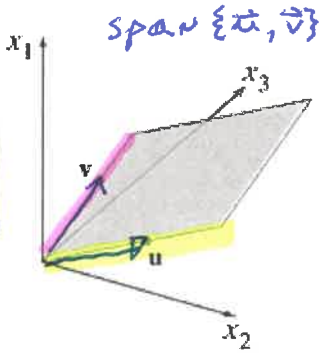
Ex 2: Describe all the solutions of the homogeneous "system".

$$3x_1 - 4x_2 + 5x_3 = 0 \Rightarrow \begin{bmatrix} 3 & -4 & 5 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -\frac{4}{3} & \frac{5}{3} & 0 \end{bmatrix}$$

The solutions form a plane. The plane represents the span$\{\vec{u}, \vec{v}\}$

$$\begin{aligned} x_1 &= \tfrac{4}{3}x_2 - \tfrac{5}{3}x_3 \\ x_2 &= x_2 \text{ (free)} \\ x_3 &= x_3 \text{ (free)} \end{aligned} \Rightarrow \vec{x} = x_2 \underbrace{\begin{bmatrix} 4/3 \\ 1 \\ 0 \end{bmatrix}}_{\vec{u}} + x_3 \underbrace{\begin{bmatrix} -5/3 \\ 0 \\ 1 \end{bmatrix}}_{\vec{v}}$$



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The previous example demonstrates how we can write solutions in Parametric Vector Form. $\mathbf{x} = s\mathbf{u} + t\mathbf{v} \qquad (s, t \in \mathbb{R})$

result of Ex1: $\vec{x} = s\vec{u}$ where $s = x_3$ and $\vec{u} = \begin{bmatrix} -17/8 \\ 3/4 \\ 1 \end{bmatrix}$

result of Ex2: $\vec{x} = s\vec{u} + t\vec{v}$ where $s = x_2$, $t = x_3$, $\vec{u} = \begin{bmatrix} 4/3 \\ 1 \\ 0 \end{bmatrix}$ and $\vec{v} =$

Solutions of Nonhomogeneous Systems

Ex 3: Describe all solutions of $A\mathbf{x} = \mathbf{b}$. $\quad A = \begin{bmatrix} 1 & 3 & 1 \\ -4 & -9 & 2 \\ 0 & -3 & -6 \end{bmatrix}$ and $b = \begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} 1 & 3 & 1 & 1 \\ -4 & -9 & 2 & -1 \\ 0 & -3 & -6 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -5 & -2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

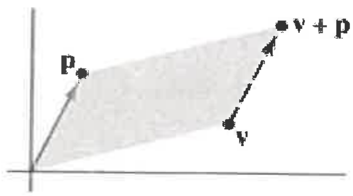
$$\Rightarrow \begin{aligned} x_1 &= -2 + 5x_3 \\ x_2 &= 1 - 2x_3 \\ x_3 &= x_3 \text{ (free)} \end{aligned}$$

$$\Rightarrow \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}$$

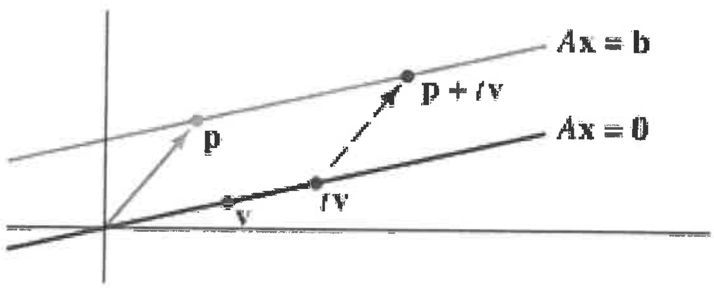
$\Rightarrow \vec{x} = \vec{p} + t\vec{v}$

where $\vec{p} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$, $t = x_3$

and $\vec{v} = \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}$



To visualize the solution set of $A\mathbf{x} = \mathbf{b}$ geometrically, we can think of vector addition as a translation.



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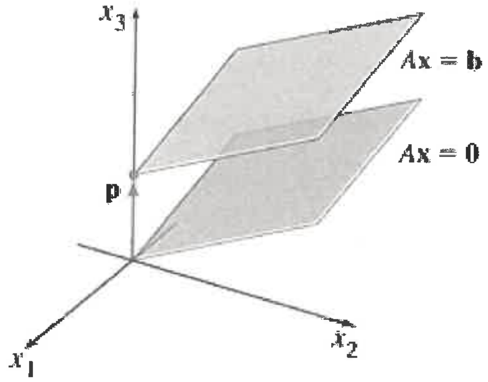
The solution set of $A\mathbf{x}=\mathbf{b}$ is a line through $\mathbf{p}$ parallel to the solution set of $A\vec{x}=\vec{0}$.

THEOREM 6

Suppose the equation $A\mathbf{x}=\mathbf{b}$ is consistent for some given $\mathbf{b}$, and let $\mathbf{p}$ be a solution. Then the solution set of $A\mathbf{x}=\mathbf{b}$ is the set of all vectors of the form $\mathbf{w}=\mathbf{p}+\mathbf{v}_h$, where $\mathbf{v}_h$ is any solution of the homogeneous equation $A\mathbf{x}=\mathbf{0}$.



claim: : Suppose that $\mathbf{p}$ is a solution of $A\mathbf{x}=\mathbf{b}$, so that $A\mathbf{p}=\mathbf{b}$. Let $\mathbf{v}_h$ be any solution to the homogeneous equation $A\mathbf{x}=\mathbf{0}$, and let $\mathbf{w}=\mathbf{p}+\mathbf{v}_h$, Show that $\mathbf{w}$ is a solution to $A\mathbf{x}=\mathbf{b}$.

proof.

Let A, $\vec{p}$, $\vec{b}$, and $\vec{v}_h$ be given as above.

$$\Rightarrow A\vec{w} = A(\vec{p}+\vec{v}_h)$$

$$= A\vec{p} + A\vec{v}_h$$

$$= \vec{b} + \vec{0}$$

$$= \vec{b}$$

$\therefore$ $\vec{w}$ is a solution to $A\vec{x}=\vec{b}$

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Writing a Solution Set (of a Consistent System) in Parametric Vector Form

1. Row reduce the augmented matrix to reduced echelon form.

2. Express each basic variable in terms of any free variables appearing in an equation.

3. Write a typical solution **x** as a vector whose entries depend on the free variables, if any.

4. Decompose **x** into a linear combination of vectors (with numeric entries) using the free variables as parameters.

Ex 4: Each of the following equations determines a plane in $\mathbb{R}^3$. Do the two planes intersect? If so, describe their intersection.

$$x_1 + 4x_2 - 5x_3 = 0$$
$$2x_1 - x_2 + 8x_3 = 9$$

$$\Rightarrow \begin{bmatrix} 1 & 4 & -5 & 0 \\ 2 & -1 & 8 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 4 \\ 0 & 1 & -2 & -1 \end{bmatrix}$$

The system is consistent so the planes intersect.

$x_1 = 4 - 3x_3$

$x_2 = -1 + 2x_3$

$x_3 = x_3$ (free)

$$\vec{x} = \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$$

line thru $(4,-1,0)$ parallel to $\langle -3,2,1 \rangle$

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They intersect along the line parameterized above.

Ex 5: Write the general solution of $10x_1 - 3x_2 - 2x_3 = 7$ in parametric vector form,

$x_1 = 0.7 - 0.3x_2 - 0.2x_3$

$x_2 = x_2$ (free)

$x_3 = x_3$ (free)

$$\Rightarrow \vec{x} = \begin{bmatrix} 0.7 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -0.3 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -0.2 \\ 0 \\ 1 \end{bmatrix}$$

connection
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plane thru $(0.7,0,0)$ that includes the vectors $\langle -0.3, 1, 0 \rangle$ and $\langle -0.2, 0, 1$

1.5: Solution Sets of Linear Systems

1.6 – Applications (read/review Network Flow as well – pages 53 – 54)

Balancing Chemical Equations

Chemical equations describe the quantities of substances consumed and produced by chemical reactions. For instance, when propane gas burns, the propane (C_3H_8) combines with oxygen (O_2) to form carbon dioxide (CO_2) and water (H_2O), according to an equation of the form

$$(x_1)C_3H_8 + (x_2)O_2 \to (x_3)CO_2 + (x_4)H_2O \qquad (4)$$

$$C_3H_8: \begin{bmatrix}3\\8\\0\end{bmatrix}, O_2: \begin{bmatrix}0\\0\\2\end{bmatrix}, CO_2: \begin{bmatrix}1\\0\\2\end{bmatrix}, H_2O: \begin{bmatrix}0\\2\\1\end{bmatrix} \begin{matrix}\leftarrow \text{Carbon}\\ \leftarrow \text{Hydrogen}\\ \leftarrow \text{Oxygen}\end{matrix}$$

$$\Rightarrow x_1\begin{bmatrix}3\\8\\0\end{bmatrix} + x_2\begin{bmatrix}0\\0\\2\end{bmatrix} = x_3\begin{bmatrix}1\\0\\2\end{bmatrix} + x_4\begin{bmatrix}0\\2\\1\end{bmatrix}$$

$$\Rightarrow x_1\begin{bmatrix}3\\8\\0\end{bmatrix} + x_2\begin{bmatrix}0\\0\\2\end{bmatrix} - x_3\begin{bmatrix}1\\0\\2\end{bmatrix} - x_4\begin{bmatrix}0\\2\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}3 & 0 & -1 & 0 & 0\\ 8 & 0 & 0 & -2 & 0\\ 0 & 2 & -2 & -1 & 0\end{bmatrix} \sim \begin{bmatrix}1 & 0 & 0 & -1/4 & 0\\ 0 & 1 & 0 & -5/4 & 0\\ 0 & 0 & 1 & -3/4 & 0\end{bmatrix}$$

$x_1 = \frac{1}{4}x_4$

$x_2 = \frac{5}{4}x_4$

$x_3 = \frac{3}{4}x_4$

$x_4 = x_4$ (free)

$$\Rightarrow \vec{x} = \begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = x_4\begin{bmatrix}1/4\\5/4\\3/4\\1\end{bmatrix}$$

this means $1C_3H_8 + 5O_2 \to 3CO_2 + 4H_2O$