Mathematics and the Loss of Certainty by Morris Kline

The attitude of the 18th and early 19th centuries was expressed by J. Hoëne-Wronski (1775–1853), who was a great algorithmist but not concerned with rigor. A paper of his was criticized by a commission of the Paris Academy of Sciences as lacking in rigor, and Wronski replied that this was "pedantry which prefers means to the end."

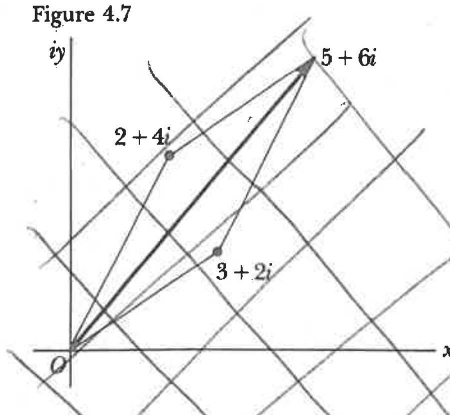
Lacroix, in the second edition (1810–19) of his three-volume *Treatise on Differential and Integral Calculus*, said in the Preface to the first volume, "Such subtleties as the Greeks worried about we no longer need." The typical attitude of the times was to ask why one should go to the trouble of proving by abstruse reasoning things one never doubts in the first place or of demonstrating what is more evident by means of what is less evident.

Even later in the 19th century, Karl Gustav Jacob Jacobi (1804–1851), who left many points in his work on elliptic functions incomplete, said, "For Gaussian rigor we have no time." Many acted as though what defied proof needed no proof. For most men rigor was not a concern. Often what they said could be rigorized by the method of Archimedes, could not have been rigorized by a modern Archimedes. This is particularly true of the work on differentiation which had no parallel in Greek mathematics. What d'Alembert said in 1743, "Up to the present . . . more concern has been given to enlarging the building than to illuminating the entrance, to raising it higher than to giving proper strength to the foundations," applies to the work of the entire 18th and early 19th centuries.

By the middle of the 19th century, the regard for proof had fallen so low that some mathematicians did not even bother to execute full proofs where they might have been able to achieve them. Arthur Cayley (1821–1895), one of the superb algebraic geometers and the inventor of what is called matrix algebra (Chapter IV), stated a theorem on matrices known as the Cayley-Hamilton theorem. A matrix is a rectangular array of numbers and in the case of square matrices there are n numbers in each row and in each column. Cayley verified that his theorem was true for 2 by 2 matrices and in a paper of 1858 said, "I have not thought it necessary to undertake the labor of a formal proof of the theorem in the general case of a matrix of any degree [n by n]."

James Joseph Sylvester (1814–1897), an excellent British algebraist, spent the years from 1876 to 1884 as a professor at Johns Hopkins University. In his lectures he would say, "I haven't proved this, but I am as sure as I can be of anything that it must be so." He would then use the result to prove new theorems. Often at the end of the next lecture he would admit that what he had been so sure of was false. In 1889 he proved a theorem about 3 by 3 matrices and merely indicated a few additional points which have to be considered to prove the theorem for n by n matrices.

Figure 4.7



as the complex number $2+4i$. The sum, $5+6i$, then represents the resultant vector OC.

This use of complex numbers to represent vectors and operations with vectors that lie in a plane had become somewhat well known by 1830. However, if several forces act on a body, these forces and their vector representations need not and generally will not lie in one plane. If for the sake of convenience we call ordinary real numbers one-dimensional numbers and complex numbers two-dimensional, then what would be needed to represent and work algebraically with vectors in space is some sort of three-dimensional number. The desired operations with these three-dimensional numbers, as in the case of complex numbers, would have to include addition, subtraction, multiplication, and division and moreover obey the usual properties of real and complex numbers so that algebraic operations could be applied freely and effectively. Hence mathematicians began a search for what was called a three-dimensional complex number and its algebra.

Many mathematicians took up this problem. The creation in 1843 of a useful spatial analogue of complex numbers is due to William R. Hamilton. For fifteen years Hamilton was baffled. All of the numbers known to mathematicians at this time possessed the commutative property of multiplication, that is $ab=ba$, and it was natural for Hamilton to believe that the three-dimensional or three-component numbers he sought should possess this same property as well as the other properties that real and complex numbers possess. Hamilton succeeded but only by making two compromises. First, his new numbers contained four components and, second, he had to sacrifice the commutative law of multiplication. Both features were revolutionary for algebra. He called the new numbers quaternions.

Whereas a complex number is of the form $a+bi$ wherein $i=\sqrt{-1}$, a quaternion is a number of the form

$$a+bi+cj+dk$$

wherein the i, j, and k possess the same property as $\sqrt{-1}$, namely,

$$i^2=j^2=k^2=-1.$$

The criterion for equality of two quaternions is that the coefficients a, b, c, and d be equal.

Two quaternions are added by the addition of the respective coefficients to form new coefficients. Thus the sum of two quaternions is itself a quaternion. To define multiplication Hamilton had to specify what the products of i and j, i and k, and j and k should be. To ensure that the product be a quaternion and to secure for his quaternions as many of the properties of real and complex numbers as possible, he arrived at

like the

$$jk=i,\ kj=-i,\ ki=j,\ ik=-j,\ ij=k,\ ji=-k.$$ cross product

These agreements mean that multiplication is not commutative. Thus if p and q are quaternions, pq does not equal qp. Division of one quaternion by another can also be effected. However, the fact that multiplication is not commutative means that to divide the quaternion p by the quaternion q one can intend to find r such that $p=qr$ or such that $p=rq$. The quotient r need not be the same in the two cases. Though quaternions did not prove to be as broadly useful as Hamilton expected, he was able to apply them to a large number of physical and geometrical problems.

The introduction of quaternions was another shock to mathematicians. Here was a physically useful algebra which failed to possess a fundamental property of all real and complex numbers, namely, that $ab=ba$.

Not long after Hamilton created quaternions, mathematicians working in other domains introduced even stranger algebras. The famous algebraic geometer Arthur Cayley (1821–1895) introduced matrices, L.A. which are square or rectangular arrays of numbers. These too are subject to the usual operations of algebra but, as in the case of quaternions, lack the commutative property of multiplication. Moreover, the product of two matrices can be zero even though neither factor is zero. Quaternions and matrices were but the forerunners of a host of new algebras with stranger and stranger properties. Hermann Günther Grassmann (1809–1877), created a variety of such algebras. These were even more general than Hamilton's quaternions. Unfortunately, Grassmann was a high-school teacher and so it took many years before his work received the attention it deserved. In any case, Grassmann's work added to the variety of new types of what are now called hypernumbers.

The creation of new algebras for special purposes did not in itself challenge the truth of ordinary arithmetic and its extensions in algebra and analysis. After all, the ordinary real and complex numbers were used for totally different purposes where their applicability seemed unquestionable. Nevertheless, the very fact that new algebras appeared on the scene made men doubt the truth of the familiar arithmetic and algebra, just as people who learn about the customs of a strange civilization begin to question their own.

The sharpest attack on the truth of arithmetic came from Hermann von Helmholtz (1821–1894), a superb physician, physicist, and mathematician. In his *Counting and Measuring* (1887) he considered the main problem in arithmetic to be the justification of the *automatic* application of arithmetic to physical phenomena. His conclusion was that only experience can tell us where the laws of arithmetic do apply. We cannot be sure a priori that they do apply in any given situation.

Helmholtz made many pertinent observations. The very concept of number is derived from experiences. Some kinds of experience suggest the usual types of number, whole numbers, fractions, and irrational numbers, and their properties. To these experiences the familiar numbers are applicable. We recognize that virtually equivalent objects exist and so we recognize that we may speak, for example, of two cows. However, these objects must not disappear or merge or divide. One raindrop added to another does not make two raindrops. Even the notion of equality cannot be applied automatically to experience. It would seem certain that if object a equals c and b equals c that a must equal b. But two pitches of sound may seem to equal a third and yet the ear might distinguish the first two. Here things equal to the same thing need not equal each other. Likewise colors a and b may seem the same as do colors b and c but a and c can be distinguished.

Many examples may be adduced to show that the naïve application of arithmetic would lead to nonsense. Thus if one mixes two equal volumes of water, one at 40° Fahrenheit and the other at 50°, one does not get two volumes at 90°. If two simple sounds, one of 100 cycles per second and another of 200 cycles per second, are superposed one does not get a sound whose frequency is 300 cycles per second. In fact the composite sound has a frequency of 100 cycles per second. If two resistances of magnitudes R_1 and R_2 are connected in parallel in an electrical circuit, their combined effective resistance is $R_1R_2/(R_1+R_2)$. Further, as Henri Lebesgue facetiously pointed out, if one puts a lion and a rabbit in a cage, one will not find two animals in the cage later on.

33 example.

We learn in chemistry that when one mixes hydrogen and oxygen he obtains water. But if someone takes two volumes of hydrogen and one volume of oxygen, he obtains, not three, but two volumes of water vapor. Likewise, one volume of nitrogen and three volumes of hydrogen yield two volumes of ammonia. We happen to know the physical explanation of these surprising arithmetic facts. By Avogadro's hypothesis, equal volumes of any gas, under the same conditions of temperature and pressure, contain the same number of *particles.* If, then, a given volume of oxygen contains 10 molecules, the same volume of hydrogen will also contain 10 molecules. Then there are 20 molecules in two volumes of hydrogen. Now it happens that the molecules of oxygen and hydrogen are diatomic; that is, each contains two atoms. Each of these 20 diatomic hydrogen *molecules* combines with one atom of the 10 molecules of oxygen to form 20 molecules of water or two volumes of water, but not three. Thus arithmetic fails to describe correctly the result of combining gases by volumes.

Ordinary arithmetic also fails to describe the combination of some liquids by volume. If a quart of gin is mixed with a quart of vermouth one does not get two quarts of the mixture but a quantity slightly less. One quart of alcohol and one quart of water yield about 1.8 quarts of vodka. This is true of most mixtures of alcoholic liquids. Three tablespoons of water and one tablespoon of salt do not make four tablespoons. Some chemical mixtures not only do not add up by volume but explode.

Not only do the properties of the whole numbers fail to apply to many physical situations, but there are practical situations in which a different arithmetic of fractions must be applied. Let us consider baseball, certainly a subject of interest to millions of Americans.

Suppose a player goes to bat 3 times in one game and 4 times in another. How many times in all did he go to bat? There is no difficulty here. He went to bat a total of 7 times. Suppose he hit the ball successfully, that is, got to first base or farther, twice in the first game and 3 times in the second. How many hits did he make in both games? Again there is no difficulty. The total number of hits is 2 + 3, or 5. However, what the audience, and the player himself, is usually most interested in is the batting average, that is, the ratio of the number of hits to the number of times at bat. In the first game this ratio was 2/3; in the second game the ratio was 3/4. And now suppose the player or a baseball fan wishes to use these two ratios to compute the batting average for both games. One would think that all he would have to do was to add the two fractions by the usual method of adding fractions. That is,

$$\frac{2}{3}+\frac{3}{4}=\frac{17}{12}.$$