Chapter 16 Summary Fall 2013

16.1 Vector Field

1) A vector field $\vec{F}$ assigns a vector $\vec{F}(x,y)$ or $\vec{F}(x,y,z)$ to each point in $\mathbb{R}^2$ or $\mathbb{R}^3$ (or a subregion).

2) May write $\vec{F}(x,y,z)$ as $\vec{F}(\vec{X})$, where $\vec{X} = \langle x, y, z\rangle$ is identified with the point (x,y,z).

3) Examples

a) Gravity: $\vec{F}(\vec{X}) = -\dfrac{mMG}{|\vec{X}|^3}\, X$

b) Electric Force Field $\vec{F}(\vec{X}) = \dfrac{\varepsilon q Q}{|\vec{X}|^3}\, \vec{X}$
(Coulomb's Laws)

c) Gradient Field $\vec{F}(x,y) = \nabla f(x,y)$
or $\vec{F}(x,y,z) = \nabla f(x,y,z)$

4) $\vec{F}(\vec{X})$ is conservative if it is a gradient field:
$\vec{F}(\vec{X}) = \nabla f(x,y)$ or $\nabla f(x,y,z)$.
f is the potential of $\vec{F}$ if $\nabla f = \vec{F}$

5) Gravitational fields are conservative
$f(\vec{X}) = f(x,y,z) = \dfrac{mMG}{\sqrt{x^2+y^2+z^2}}$

Not conservative, eg: $\vec{F}(x,y) = \langle x^2+y,\ x^2y\rangle = \langle P, Q$
Note $P_y \neq Q_x$

16.2 Line Integrals

1) Want to integrate $f(x,y)$ along curve C given by $\vec{r}(t) = \langle x(t), y(t) \rangle$, where C is
? → smooth.

2) $\int_C f(x,y)\,ds = \lim_{n\to\infty} \sum f(x_i^*, y_i^*)\,\Delta s_i$,
(x_i^*, y_i^*) a pt. on i^{th} sub-arc.

3) Fact: if C is $\vec{r}(t) = \langle x(t), y(t) \rangle$, $a \le t \le b$, then

$$\int_C f(x,y)\,ds = \int_a^b f(x(t),y(t))\sqrt{(x'(t))^2 + (y'(t))^2}\,dt$$

since $ds = \sqrt{(x'(t))^2 + (y'(t))^2}\,dt$.

4) Notes

? → a) Value of integral is independent of parameterizat but we assume C is traversed only once.

b) $f(x,y) \ge 0$ on $C \Rightarrow \int_C f(x,y)\,ds =$ "area under f, above

c) $\rho(x,y) =$ linear density of wire at (x,y), then $\int_C \rho(x,y)\,ds =$ mass of wire

5) Define line integrals of f along C, with respect to x and y.

$$\int_C f(x,y)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\,\Delta x_i \text{ ; etc. for } dy.$$

? → To calculate: $\int_C f(x,y)\,dy = \int_a^b f(x(t),y(t))\cdot y'(t)\,dt$

$$\int_C f(x,y)\,dx = \int_a^b f(x(t),y(t))\,x'(t)\,dt$$

since $x = x(t)$, $y = y(t)$, so $dx = x'(t)\,dt$, $dy = y'(t)\,dt$.

6) Often see $\int_C P(x,y)\,dx + \int_C Q(x,y)\,dy$; for convenience we write this $\int_C P\,dx + Q\,dy$

7) If $C = C_1 + \dots + C_k$, each C_i smooth, then C is p.w. smooth and $\int_C f(x,y)\,ds = \sum_{i=1}^{n} \int_{C_i} f(x,y)\,ds$, by defn.

8) Can extend results to $\mathbb{R}^3$

$$\int_C f(x,y,z)\,dS = \int_a f(x(t),y(t),z(t))\sqrt{(x'(t))^2+(y'(t))^2+(z'(t))^2}\,dt$$

(or dx, dy or dz also)

9) Note: $\int_C f(x,y)\,ds = \int_a^b f(\vec{r}(t))\,|\vec{r}\,'(t)|\,dt$

and $\int_C f(x,y,z)\,dS = \int_a^b f(\vec{r}(t))\,|\vec{r}\,'(t)|\,dt$

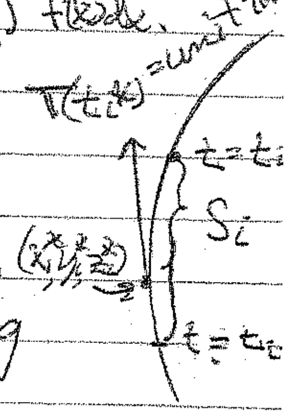
16.2

10) Line Integrals over Vector fields

a) Work: $W = \vec{F} \cdot \vec{D}$ $\quad \vec{F}$ $\quad \vec{D}$ = displacement

not needed - just parallel result?

If force $f(x)$ moves object from a to b along a line (force in direction of motion), work $= \int_a^b f(x)dx$.



b) If C is smooth for $a \le t \le b$, and
$\vec{F} = P(x,y,z)\vec{i} + Q(x,y,z)\vec{j} + R(x,y,z)\vec{k}$ is a continuous force field in $\mathbb{R}^3$ (or $\mathbb{R}^2$),

? → the work done moving an object along segment S_i of length Δs is approx.

$$\underbrace{\vec{F}(x_i^*, y_i^*, z_i^*)}_{F} \cdot \underbrace{\vec{T}_i(t_i^*)\,\Delta s}_{D}$$

Resulting integral is

$$W = \int_C F(x,y,z) \cdot T(x,y,z)\,ds = \int_C F \cdot T\,ds$$

$$= \int_a^b \vec{F}(\vec{r}(t)) \cdot \frac{\vec{r}'(t)}{|\vec{r}'(t)|}\,|r'(t)|\,dt = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\,dt$$

$$= \int_C \vec{F} \cdot d\vec{r} = \text{line integral of } \vec{F} \text{ along } C.$$

Also: if $\vec{F} = \langle P(x,y), Q(x,y) \rangle$, then

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \left(P(x,y)\,x'(t) + Q(x,y)\,y'(t)\right)dt$$

$$= \int_C P\,dx + Q\,dy$$

16.3 Fundamental Theorem of Line Integrals.

1) Fund. Thm of line integrals:

Suppose C is smooth, given by $r(t)$, $a \le t \le b$.
If f is a diff. function of two or three variables, and ∇f is continuous on C, then

$$\int_C \nabla f \cdot dr = f(r(b)) - f(r(a))$$

2) Notes

a) So if F is conserv., $\int_a^b \vec{F} \cdot d\vec{r} = f(r(b)) - f(r(a))$.
f = potential of F.

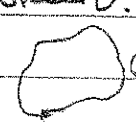
b) Thm is true if C is p.w. smooth.

c) Proof is easy.

3) Independence of Path.

If $\vec{F}$ is continuous with domain D, we say $\int_C \vec{F} \cdot d\vec{r}$ is independent of path in D if given any two points A, B in D, then
$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}$, for any two paths C_1 and C_2 connecting A to B.

Note: line integrals of conservative vector fields are always indept. of path. Conserv. $\Rightarrow$ indept of pa



C.

4) Defn: C is closed if $\vec{r}(b) = \vec{r}(a)$

Properties: a) If C is closed and $\vec{F}$ is indept. of path then $\int_C F \cdot dr = 0$

b) If $\int_C F \cdot dr = 0$ for every closed path in domain D of $\vec{F}$, then $\vec{F}$ is indept. of path on D.

So c) $\int_C F\, dr$ is indept of path $\Longleftrightarrow$ $\int F\, dr = 0$ for any closed pa

16.3

5) Terminology

a) region D is open if every point is interior.

b) D is connected if any two points in D lie on a common curve which entirely within D.

6) We know conservative $\Rightarrow$ indept. of path

Now, under certain conditions, converse is true.

Thrm: If $\vec{F}$ is continuous on open, connected region D, then if F is indept. of path on D, then F is conservative, ie $F = \nabla f$ for some f.

why important? $\longrightarrow$ So, given open connected D, F conservative $\Leftrightarrow$ F indept of path on D.

7) How to identify conservative vector fields.

a) $F = \nabla f = \langle P, Q \rangle \Rightarrow \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$, by Clairaut

So if $\partial P/\partial y \neq \partial Q/\partial x$, F is <u>not</u> conservative.

But: if $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, is F conservative?

8) Defn: a) A simple closed curve does not intersect itself.

b) D is simply connected if every simple closed curve in D encloses only points in D (no holes).

9) Thrm: Suppose $\vec{F}(x,y) = \langle P(x,y), Q(x,y) \rangle$ is defined on an open simply connected region D. If P, Q have contin. 1st partials and $\partial P/\partial y = \partial Q/\partial x$, then F is conservative.

Notes: does not say how to find potential of F.

Case for $\mathbb{R}^3$ comes later.

16.3

10.) Can use partial integration (ala Diff Eqns) to find potential $f(x,y)$, given $\nabla f = \vec{F} = \langle P, Q \rangle$.
Can also do this if $\nabla f(x,y,z) = \vec{F} = \langle P, Q, R \rangle$.

16.4 Green's Thrm

1) Defn: Simple closed curve C has positive orientation if C is traversed once in the counter-clockwise direction

2) Note: Recall $\int_C \vec{F}\cdot d\vec{r} = \int_C P\,dx + Q\,dy$ (in $\mathbb{R}^2$).

3) Green's Theorem

Suppose a simple, closed positively-oriented p.w. smooth curve C bounds a region D in the plane.

If P, Q have continuous partials on an open region containing D, then

$$\int_C P\,dx + Q\,dy \left(= \int_C \vec{F}\,d\vec{r}\right) = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA.$$

4) Notes a) $\int_C P\,dx + Q\,dy$ is often written $\oint_C$ or $\int_{\partial D}$

b) Green's Thrm is analog of Fund Thrm of Calc: $\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_C P\,dx + Q\,dy$, involves anti derivatives on R.H.S., and C is boundary of D

c) If we want area of D, choose P & Q such that $\partial Q/\partial x - \partial P/\partial y = 1$ (see below), so $\iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA = \iint_D 1\,dA$ and $\int_C P\,dx + Q\,dy$ will give area.

$P(x,y) = -y$ or $P(x,y) = -\frac{1}{2}y$ etc.
$Q(x,y) = 0$ $\quad$ $Q(x,y) = \frac{1}{2}x$

d) Pos. Oriented $\Rightarrow$ region is on the left as we travel

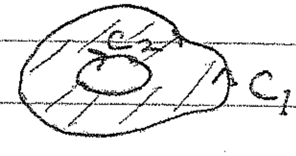
e) In $\displaystyle\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \oint_C P\,dx + Q\,dy$,

we trade one type of integral for the other.
Given $\displaystyle\oint_C \vec{F}\cdot d\vec{r} = \int P\,dx + Q\,dy$, $\displaystyle\iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,d$

may be easier. When starting with
$\displaystyle\iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA$, often it is area of D we want

and we can choose Q and P so $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$,
and then switch to $\displaystyle\oint_C P\,dx + Q\,dy$.

f) If region D has a hole (not simply connecte
can still use Green's Thm:



$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \oint_{\text{boundary of } D} P\,dx + Q\,dy$$

$$= \oint_{C_1} P\,dx + Q\,dy + \oint_{C_2} P\,dx + Q\,dy$$

16.5 Curl + Divergence

1) Defn: If $\vec{F} = \langle P, Q, R \rangle$ and the 1^{st} partials of P, Q + R all exist, then

$$\text{curl } \vec{F} = \left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, -\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right), \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle$$

Alternately, if ∇ = del operator is defined by

$$\nabla = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}, \text{ and}$$

$$\nabla(f) = \nabla f = \langle f_x, f_y, f_z \rangle, \text{ then}$$

$$\text{Curl } \vec{F} = \nabla \times F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix}$$

2) Divergence

Defn: If $\vec{F} = \langle P, Q, R \rangle$, the divergence of $\vec{F}$ is defined by

$$\text{div } F = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

We write $\text{div } F = \nabla \cdot \vec{F}$

3) Thrm: $\text{div}(\text{curl } F) = 0$ for all F.

Proof: easy computation.

4) Thrm If $\vec{F}$ is conservative, then $\text{curl}(\vec{F}) = \vec{0}$

Proof: easy.

5) Thrm If $\vec{F} = \langle P, Q, R \rangle$ has domain $\mathbb{R}^3$ and P, Q + R have contin. 1^{st} partials, then

$$\text{Curl}(\vec{F}) = \vec{0} \iff F \text{ is conservative.}$$

6) Notes: a) "$\mathrm{curl}(\vec{F}) = \vec{0} \Rightarrow F$ is conservative" is the 3-D equivalent of "$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0 \Rightarrow F$ is conserv."

b) If $\vec{F} = \nabla f$ is conservative, then

$$\mathrm{div}(\nabla f) = \nabla \cdot \nabla f = "\nabla^2 f" = f_{xx} + f_{yy} + f_{zz}$$

* ∇^2 is the LaPlace operator, and the LaPlace equation is $\nabla^2 f = 0$

or $f_{xx} + f_{yy} + f_{zz} = 0$ Solutions play role in fluid heat flo

7) Vector Form of Green's Therm.

If $\vec{F} = \langle P, Q \rangle$, then in R^3, $\vec{F} = \langle P, Q, 0 \rangle$.

$\mathrm{curl}\,\vec{F} = \langle 0, 0, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \rangle = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k} =$

So $\oint_C \vec{F}\,d\vec{r} = \iint_D \mathrm{curl}(\vec{F}) \cdot \vec{k}\,dA$, by Green

Using $\oint_C \vec{F} \cdot d\vec{r} = \int_C \vec{F} \cdot \vec{T}\,ds$, we have

$\oint_C \vec{F} \cdot \vec{T}\,ds = \iint_D \mathrm{curl}\,\vec{F} \cdot \vec{k}\,dA$. This is the line integral of the tangential component of $\vec{F}$

It can be shown that if $\vec{n}$ is normal to the path C (and to $\vec{T}$ and $\vec{r}'(t)$), then

$$\oint_C \vec{F} \cdot \vec{n}\,ds = \iint_D \mathrm{div}\,\vec{F}\,dA.$$

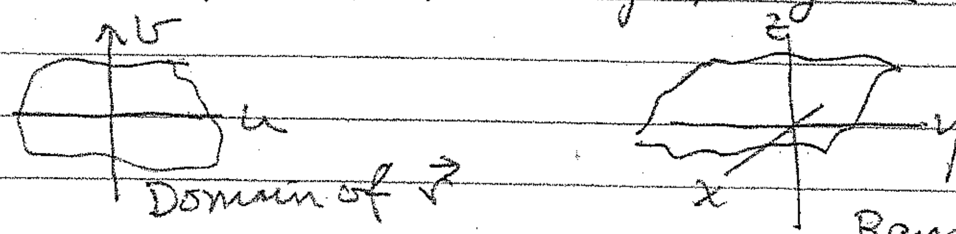
$\oint_C \vec{F} \cdot \vec{n}\,ds$ is the normal component of $\vec{F}$ along

16.6 Parametric Surfaces

1) Note $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ is a 1-dim parametric graph in $\mathbb{R}^3$ (one variable t).

2) A parametric surface can be defined by

$$\vec{r}(u,v) = x(u,v)\,\vec{i} + y(u,v)\,\vec{j} + z(u,v)\,\vec{k}$$



Domain of $\vec{r}$

Range of $\vec{r}$
surface

3) Examples.

a) $r(\theta,\phi) = \langle a\cos\theta\sin\phi, a\sin\theta\sin\phi, a\cos\phi \rangle$
Sphere, center origin, radius a (fixed).

b) Cylinder on z axis, radius r_0.
$x = 5\cos(u)$, $y = 5\sin(u)$, $z = v$.
$r(u,v) = \langle 5\cos(u), 5\sin(u), v \rangle$

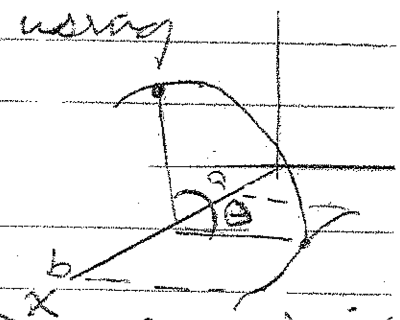
c) Plane through $P_0 = \vec{r}_0$, containing vectors $\vec{a}, \vec{b}$ (parallel to both $\vec{a}, \vec{b}$.)
$\vec{r}(u,v) = \vec{r}_0 + u\vec{a} + v\vec{b}$ $\quad P_0 = (x_0, y_0, z_0)$, $\vec{u} = \langle u_1, u_2 \ldots$, $\vec{v} = \langle v_1, v_2, v_3 \rangle$
$= \langle x_0 + ua_1 + vb_1, y_0 + ua_2 + vb_2, z_0 + ua_3 + vb_3 \rangle$

d) Elliptic paraboloid $y = ax^2 + by^2$
could use: $x = x$, $z = z$, $y = ax^2 + by^2$, i.e., $\vec{r}(x,z) = \langle x, ax^2 + by^2, z \rangle$
or $\vec{r}(u,v) = \langle u, au^2 + bv^2, v \rangle$

e) In general, if $z = f(x,y)$ defines a surface S, then S is also parameterized by $r(u,v) = \langle u, v, f(u,v) \rangle$. (just as $\begin{matrix} x=t \\ y=f(t) \end{matrix}$ will parameterize $y=f(x)$)

f) Top half of cone $z^2 = x^2+y^2$, ie, $z = \sqrt{x^2+y^2}$
could use $r(u,v) = \langle u, v, \sqrt{u^2+v^2} \rangle$
or $\vec{r}(r,\theta) = \langle r\cos\theta, r\sin(\theta), r \rangle$ pol
or $r(\rho,\theta) = \langle \rho\cos(\theta)\sin(\phi_0), \rho\sin(\theta)\sin(\phi_0), \rho\cos$
where ϕ_0 is fixed.

g) If $y = f(x)$, $a \le x \le b$, is revolved about x axis to generate surface of revolution, we can parameterize the surface using
$x = x$
$y = f(x)\cos\theta$
$z = f(x)\sin(\theta)$



$r\langle x,\theta \rangle = \langle x, f(x)\cos\theta, f(x)\sin(\theta) \rangle$

θ = angle wi the pos. y axis

h) Grid curves. If S is defined by $\vec{r}(u,v)$ for $(u,v) \in D$ = domain, then holding one of u or constant and graphing $r(u,v)$ gives a grid curve o

Ex:

16.6

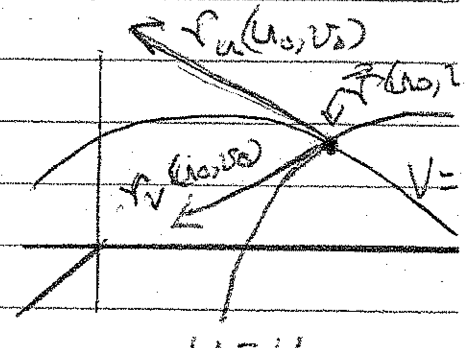
4) Tangent Planes

If $\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$ gives S we define

$$\frac{\partial \vec{r}}{\partial u} = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle . \text{ Likewise } \frac{\partial \vec{r}}{\partial v}$$

a) The vectors $\vec{r}_u(u,v)$ and $\vec{r}_v(u,v)$ will be tangent to the grid curves at any particular point, and tangent to S at that point.

b) The tangent plane at $P = \vec{r}(u_0, v_0)$ on S is the plane defined by the point P and the normal vector $\vec{r}_u(u_0,v_0) \times \vec{r}_v(u_0,v_0) = \vec{n}$.



5) Surface Area

Recall: area of parallelogram determined by $\vec{a}$ and $\vec{b}$ is $|\vec{a} \times \vec{b}|$.

If $\vec{r}(u,v)$ is a vector-valued function with domain D, surface S, and S_{ij} is the patch element on S corresponding to subrectangle R_{ij} in D then area of S_{ij} can be approx. by area of parallelogram determined by $\Delta u\, r_u^*$ and which is $|\Delta u\, r_{u_{ij}}^* \times \Delta v\, r_{v_{ij}}^*|$

The surface area of S is defined to be

$$\iint_D |\vec{r}_u \times \vec{r}_v|\, dA = \lim_{\Delta u, \Delta v \to 0} \sum\sum |\Delta u\, r_{u_{ij}}^* \times \Delta v\, r_{u_{ij}}^*|$$

Note $\iint_D |\vec{r}_u \times \vec{r}_v|\, dA = \iint_D dS$, where $dS = |\vec{r}_u \times \vec{r}_v|\, dA$.

6) Special case: if S is given by $z = f(x,y)$,
then $\vec{r}(u,v) = \langle u, v, f(u,v) \rangle$, $r_u = \langle 1, 0, f_u \rangle$, $r_v = \langle 0, 1, f_v \rangle$
and $\iint_D |\vec{r}_u \times \vec{r}_v| \, dA = \iint \sqrt{f_u^2 + f_v^2 + 1} \, dA$

$= \iint_D \sqrt{f_x^2 + f_y^2 + 1} \, dA$

16.7 Surface Integrals. $dS = |r_u \times r_v|\, dA$.

$\iint dS$ = surface area.

1) If $f(x,y,z)$ is defined on S given by $\vec{r}(u,v)$, the integral of f over S is defined by

$$\lim_{m,n\to\infty} \sum_{i=1}^{n} \sum_{j=1}^{m} f(P_{ij}^*)\, \Delta S_{ij} = \iint_S f(x,y,z)\, dS$$

where $f(x,y,z) = f(r(u,v))$.

$$\text{So } \iint_S f(x,y,z)\, dS = \iint_D f(r(u,v))\, |r_u \times r_v|\, dA$$

2) Notes

a) $f(x,y,z) = 1 \Rightarrow \iint_S 1\, dS$ = surface area of S.

✓ b) If $\rho(x,y,z)$ = density/unit area, then mass of $S = \iint \rho(x,y,z)\, dS = m$,

$M_{yz} = \iint x\,\rho(x,y,z)\, dS$, $M_{xz} = \iint y\,\rho(x,y,z)\, dS$, etc

c) In $\iint_S f(x,y,z)\, dS = \iint_D f(\vec{r}(u,v))\, |r_u \times r_v|\, dA$

we need to remember to evaluate f on S, ie, replace x,y,z by $x(u,v)$, $y(u,v)$ and $z(u,v)$, resp. (this is what $f(\vec{r}(u,v))$ means).

✓ d) If S is defined by $z = g(x,y)$, then as before $|\vec{r}_u \times \vec{r}_v|$ becomes $\sqrt{g_x^2 + g_y^2 + 1}$, so

$$\iint_S f(x,y,z)\, dS = \iint_D f(x, y, g(x,y))\sqrt{1 + g_x^2 + g_y^2}\, dA$$

3) Oriented Surfaces

An orientable surface has two sides (not a Möbius Strip).

An oriented surface is an orientable surface S one which has a tangent plane at each (non-boundary) point and we can find a unit normal that varies continuously over S.

The choice of $\vec{n}$ (there is one for each side) determines the orientation.

4) Notes:

a) For $z = g(x,y)$ defining S, $\langle -g_x, -g_y, 1\rangle$ is normal to S (so is $\langle g_x, g_y, -1\rangle$), and the unit normal pointing upward is $\dfrac{\langle -g_x, -g_y, 1\rangle}{\sqrt{g_x^2 + g_y^2 + 1}}$

This is just $\dfrac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$, if $\vec{r}(u,v) = \langle u, v, g(u,v)\rangle$ is the parametric form of S.

b) In general, $\dfrac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$ is a unit normal to S; another is $-\dfrac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$.

c) Sometimes there are easier, obvious choices of the unit normal.

8) Surface Integrals of Vector Fields

Assume S = oriented surface

$\vec{n}$ = unit normal pointing positive direct

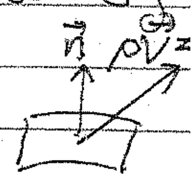
motivating example

Let $\vec{V}$ be a velocity field for fluid flowing thru

δ = density of fluid

The mass of the fluid crossing thru patch S_{ij} would be $\delta \vec{V} \cdot \vec{n}\, \Delta S_{ij}$.

$= comp_{\vec{n}}(\delta \vec{V})\ \Delta S_{ij} = comp_{\vec{n}}(\vec{F})\ \Delta S_{ij}$



Total mass thru S, per unit time, is $\iint_S \vec{F} \cdot n\, dS$.

In General: If $\vec{F}$ is any continuous vector field defined on oriented surface S, with unit normal $\vec{n}$ then the surface integral of $\vec{F}$ over S is

$$\int_S F \cdot \vec{n}\, dS, \text{ where } dS = |r_u \times r_v|\, dA$$

Notes: a) Often write $\int_S \vec{F} \cdot \vec{n}\, dS$ as $\int \vec{F} \cdot d\vec{S}$, where $d\vec{S} = \vec{n}\, dS$

b) $\int \vec{F} \cdot d\vec{S}$ is the flux of $\vec{F}$ over S.

c) Calculation form: $\int_S \vec{F} \cdot \vec{n}\, dS = \int_S F \cdot \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|} |r_u \times r_v|\, dA$
$= \int_S \vec{F} \cdot (\vec{r}_u \times \vec{r}_v)\, dA$

d) If $\vec{r}_u \times \vec{r}_v$ has wrong direction, use $\vec{r}_v \times \vec{r}_u$ (or put − sign)

e) In $\int_S \vec{F} \cdot d\vec{S}$, $\vec{F}$ is really $\vec{F}(\vec{r}(u,v))$ – must remember to evaluate $\vec{F}$ on S.

Notes

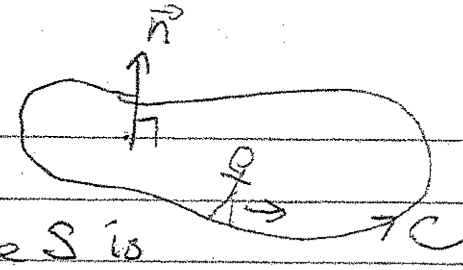
f) If S is given by $z = g(x,y)$, and $\vec{F} = \langle P, Q, R \rangle$

then $\iint \vec{F} \cdot d\vec{S} = \iint F \cdot (\vec{r}_u \times \vec{r}_v)\, dA = \iint \langle P, Q, R \rangle \cdot \langle -g_x, -g_y, 1 \rangle\, dA$

$= \iint (-Pg_x - Qg_y + R)\, dA$ assuming $\hat{k}$ is pos. direction

Using this form, we replace (in P, Q and R) x by x, y by y and z by $g(x,y)$.

16.8 Stokes Thrm



1) The boundary C of a surface S is (oriented) positively oriented if traveling C in the positive direction with your head pointed in the direction of $\vec{n}$, the surface is on your left.

2) Stokes's Theorem: Suppose S is an oriented piece-wise smooth surface, bounded by a simple closed piecewise smooth positively oriented curve C.

If $\vec{F} = \langle P, Q, R \rangle$ is a vector field and P, Q, R have continuous partials on an open region of $\mathbb{R}^3$ containing S, then

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl}\,\vec{F} \cdot d\vec{S}$$

3) Notes

a) Stokes is a higher dimensional version of Green's Theorem: $\oint_C \vec{F} \cdot d\vec{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$ where D = S is flat (in $\mathbb{R}^2$), and $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \text{curl}\, F$, with $F = \langle P, Q,$

b) $\oint_C \vec{F} \cdot d\vec{r} = \oint_C \vec{F} \cdot \vec{T}\, ds = \iint_S \text{curl}\, F \cdot n\, dS = \iint \text{curl}\, \vec{F} \cdot$

So: line integral around the boundary of S of the tangential component of F equals the surface integral over S of the normal component of curl $\vec{F}$

c) If two like-oriented surfaces S_1 and S_2 have a common positively-oriented boundary curve C, then $\iint_{S_1} \text{curl}\,\vec{F} \cdot d\vec{S} = \iint_{S_2} \vec{F} \cdot d\vec{S}$, since they both equal $\oint_C \vec{F} \cdot d\vec{r}$

16.9 The Divergence Theorem

1) Recall vector version of Green's Thm

$$\int_C \vec{F}\cdot\vec{n}\,ds = \iint_D \operatorname{div}(F)\,dA$$ if C is the boundary of D.

2) Note: A simple solid region E is simultaneously of type I, II and III (bounded by $h_1(x,y)$, $h_2(x,y)$, $h_3(x,z)$, $h_4(x,z)$, $h_5(y,z)$ and $h_6(y,z)$.)

3) Divergence Theorem

Suppose E is a simple solid region with boundary surface S having outward orientation. If $\vec{F}$ has components w/ continuous 1st partials on an open region of $\mathbb{R}^3$ containing E, then

$$\iint_S \vec{F}\cdot d\vec{S} = \iiint_E \operatorname{div}(\vec{F})\,dV$$

(So the flux of $\vec{F}$ through S = triple integral of $\operatorname{div}\vec{F}$ through E.

3) Why is divergence theorem reasonable?

For a closed surface S, if there is no expansion/contraction inside (in E), then flow in = flow out, and $\iint_S \vec{F}\,d\vec{S} = 0$. (and $\operatorname{div} F$ (in E) $= 0$).

Any net change in flow is because of expansion/contraction, measured by $\operatorname{div} F$ and $\iiint_E \operatorname{div} F$