16.6: Parametric Surfaces and Their Areas

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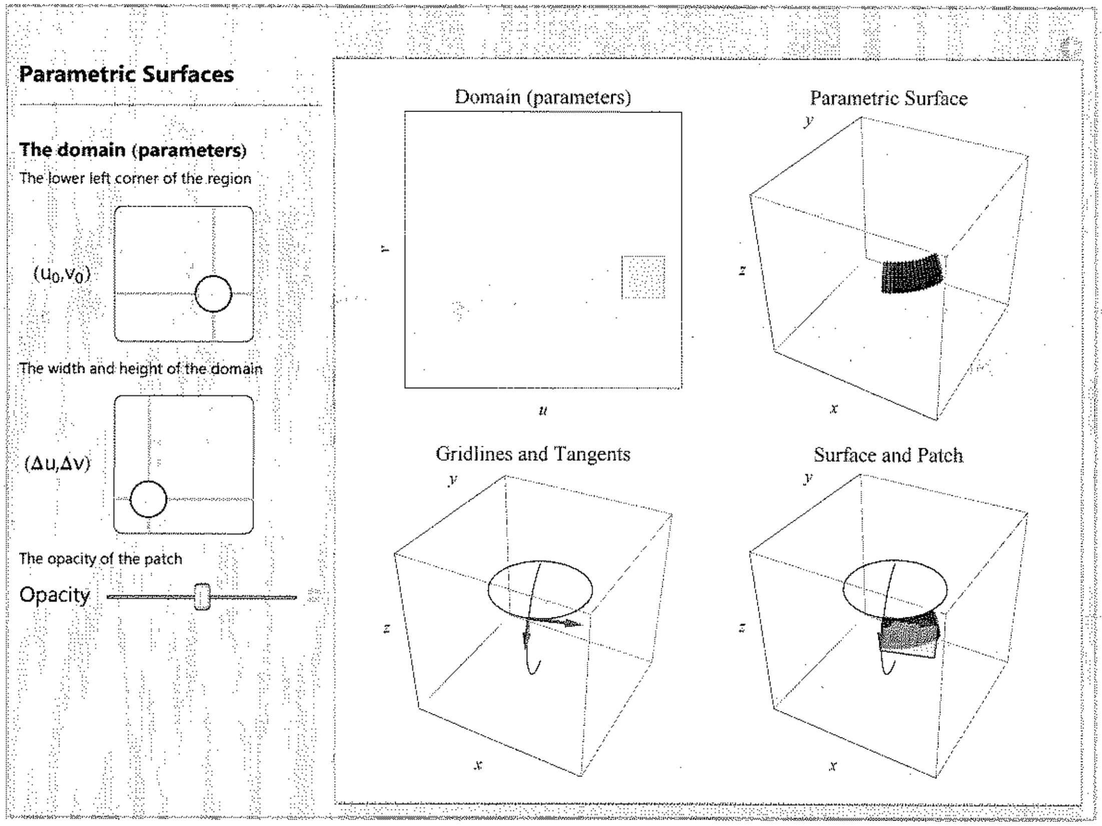

(1)

(2)

overview

(A) Graphing parametric surfaces
- show (1) on manipulate.
- 40 min

(B) Tangent planes.
- explain w/ (2) on manipulate
- skip ex 7.
- do ex 8

(C) Surface area
- explain w/ (2) on manipulate
- formula
- ex 9
- motivation that starts on p5.

16.6: Parametric Surfaces & Their Areas

To date: $\vec{r}(u) = \langle x(u), y(u), z(u) \rangle$

Now $\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$

Ex1: Basic Fcts. $z = f(x,y)$.

Ex2: Great wall: $\vec{r}(u,v) = \langle x(u), y(u), v\, f(x(u), y(u)) \rangle$

Ex3: Surfaces of revolution.

Ex4: Line in $\mathbb{R}^3$

Find parametric Eqts

Ex5: The part of $y^2 + z^2 = 16$ where $0 \le x \le 5$

Ex6: $x^2 + y^2 - z^2 = 1$ & $y \ge 0$

if $z = t$

$x = \sqrt{1+t^2}\cos(\theta)$

$y = \sqrt{1+t^2}\sin(\theta)$, $0 \le \theta \le \pi$

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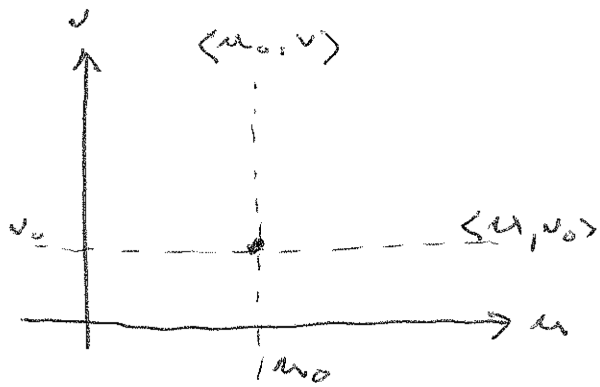
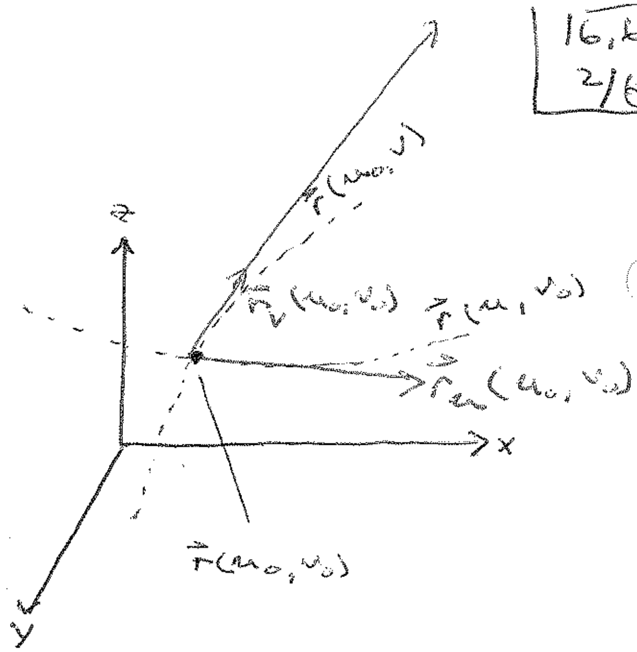
Tangent Planes





$$\vec{r}(u,v) = \langle x(u,v),\ y(u,v),\ z(u,v) \rangle$$

$$\vec{r}_u(u_0,v_0) = \langle x_u(u_0,v_0),\ y_u(u_0,v_0),\ z_u(u_0,v_0) \rangle$$

$$\vec{r}_v(u_0,v_0) = \left\langle \frac{\partial x}{\partial v},\ \frac{\partial y}{\partial v},\ \frac{\partial z}{\partial v} \right\rangle \Bigg|_{(u,v)=(u_0,v_0)}$$

Ex7: Find an eqt of the tangent plane to $\vec{r}(u,v) = \langle uv,\ u\sin v,\ v\cos u \rangle$ at $(0, \pi)$

(B)
$$\vec{r}_u(u,v) = \langle v,\ \sin v,\ -v\sin u \rangle$$
$$\& \quad \vec{r}_u(0,\pi) = \langle \pi, 0, 0 \rangle$$
$$\vec{r}_v(u,v) = \langle u,\ u\cos v,\ \cos u \rangle$$
$$\& \quad \vec{r}_v(0,\pi) = \langle 0, 0, 1 \rangle$$

vectors on the plane

(A) $\vec{r}(0,\pi) = \langle 0, 0, \pi \rangle$ ⟵ pt on the plane

(C) $$\langle \pi,0,0 \rangle \times \langle 0,0,1 \rangle = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \pi & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -\pi \vec{j}$$

is normal to the plane.

(D) $0(x-0) - \pi(y-0) + 0(z-\pi) = 0$ (plane)

$\vec{r}_u(1,1) = \langle 2u, 0, v\rangle\big|_{(1,1)} = \langle 2,0,1\rangle$

$\vec{r}_v(1,1) = \langle 0, 2v, u\rangle\big|_{(1,1)} = \langle 0,2,1\rangle$

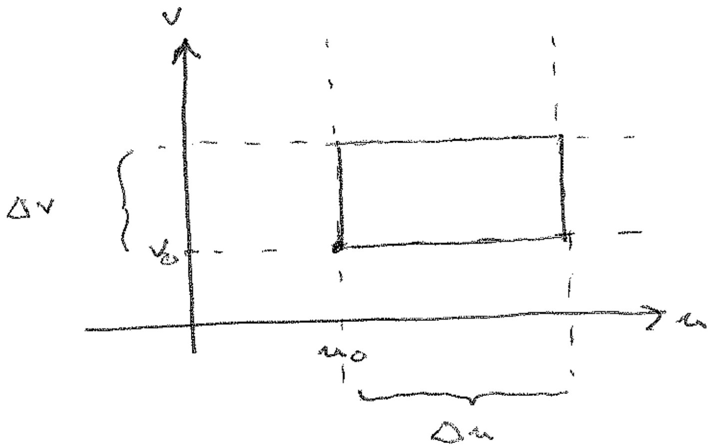
Ex 8: A more interesting example is finding the tangent plane when $(u,v)=(1,1)$ to

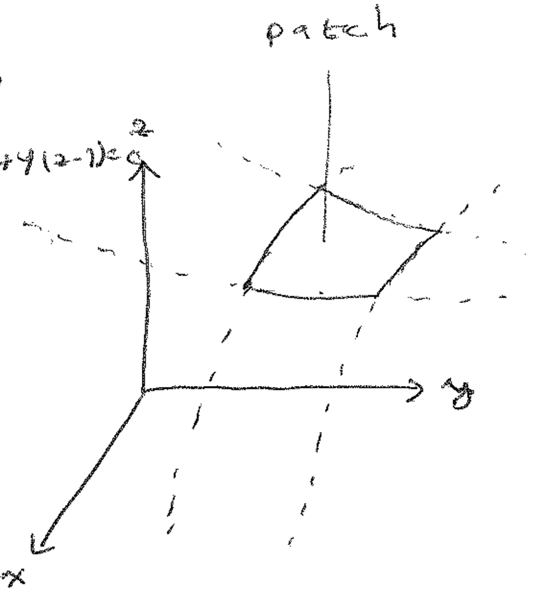
$$\vec{r}(u,v) = \langle u^2, v^2, uv\rangle$$

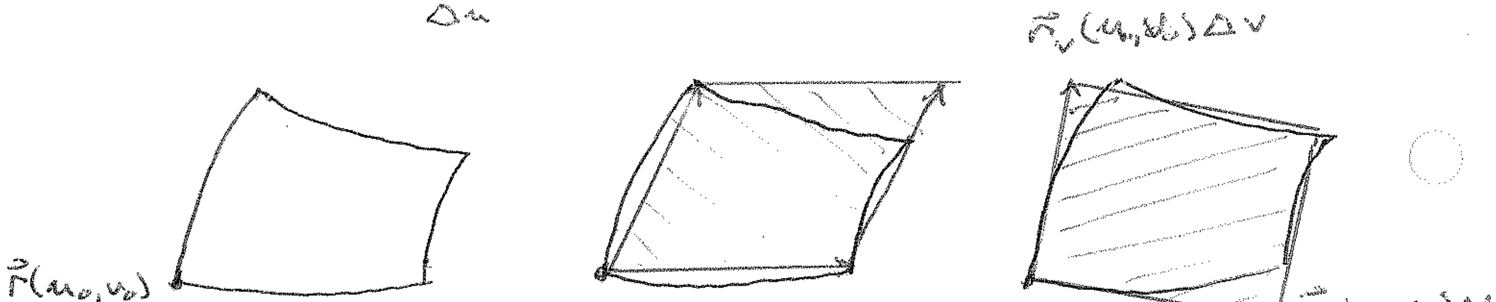
$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 0 & 1 \\ 0 & 2 & 1 \end{vmatrix} = \langle -2,-2,4\rangle$$

$-2(x-1) + -2(y-1) + 4(z-1) = 0$

Surface Area







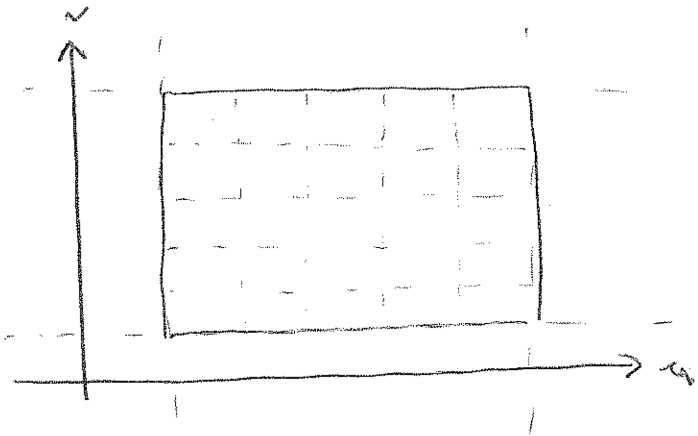
A = exact area of the patch

A ≈ area of the "secant patch"

A ≈ area of the "tangent patch"

so, the area of our patch is

$$A \approx \left|\vec{r}_u(u_0,v_0)\Delta u \times \vec{r}_v(u_0,v_0)\Delta v\right|$$
$$= \left|\vec{r}_u(u_0,v_0) \times \vec{r}_v(u_0,v_0)\right| \Delta u\, \Delta v.$$

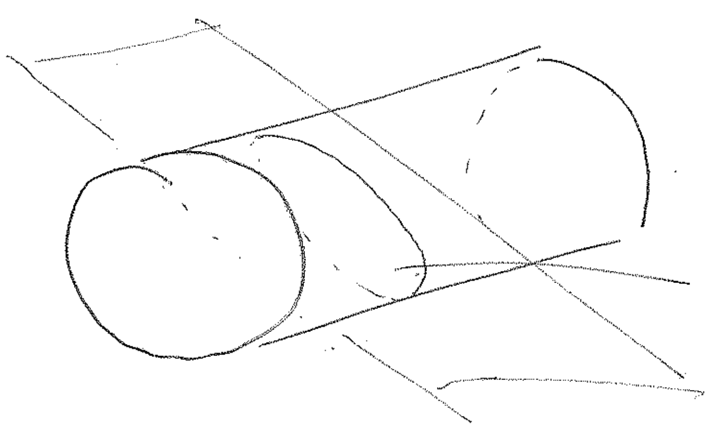


$$A \approx \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} |\vec{r}_u(u_i, v_j) \times \vec{r}_v(u_i, v_j)| \Delta u \Delta v.$$

AND if $(\Delta u, \Delta v) \longmapsto (0,0)$

$$A = \iint_D |\vec{r}_u \times \vec{r}_v| \, dA$$

<u>Ex</u> 9: Find the area of $2x + 5y + z = 10$ inside $x^2 + y^2 = 9$



$z = 10 - 2x - 5y$

$u = x$
$v = y$

area of an ellipse.

$\vec{r}(u,v) = \langle u, v, 10 - 2u - 5v \rangle$

$\vec{r}_u = \langle 1, 0, -2 \rangle$ & $\vec{r}_v = \langle 0, 1, -5 \rangle$

$$|\vec{r}_u \times \vec{r}_v| = \left\| \begin{vmatrix} i & j & k \\ 1 & 0 & -2 \\ 0 & 1 & -5 \end{vmatrix} \right\| = |\langle 2, 5, 1 \rangle| = \sqrt{30}$$

$$A = \iint_D \sqrt{30} \, dA = \sqrt{30} \cdot \pi \cdot 3^2 = 9\sqrt{30}\pi$$

Notice that this is an example of finding the SA of the graph of a fct $z = f(x,y)$.

$$\vec{r}(x,y) = \langle x, y, f(x,y) \rangle$$

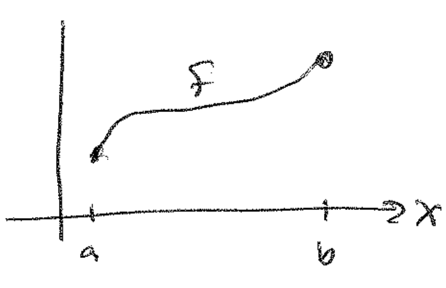
AND
$$\vec{r}_x \times \vec{r}_y = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix}$$

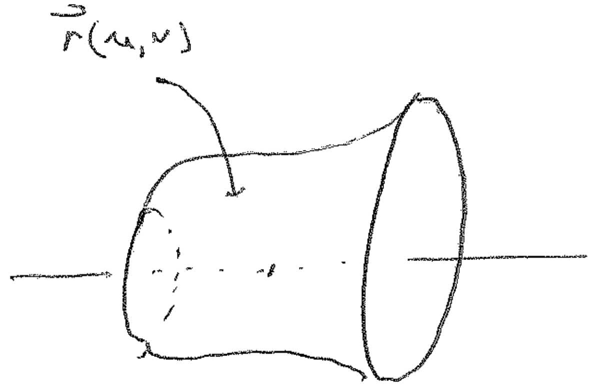
$$= \langle f_x , f_y , 1 \rangle$$

$$\Rightarrow \quad |\vec{r}_x \times \vec{r}_y| = \sqrt{1 + (f_x)^2 + (f_y)^2}$$

so
$$A = \iint_D \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\, dA .$$

Finally, what is SA of the surface formed by rotating $y = f(x)$ about the x-axis on $a \leq x \leq b$.





$$\vec{r}(u,\theta) = \langle u , f(u)\cos\theta , f(u)\sin\theta \rangle$$

where $a \leq u \leq b$ & $0 \leq \theta \leq 2\pi$

$r_u = \langle 1, f_u \cos\theta, f_u \sin\theta \rangle$

$r_\theta = \langle 0, -f \sin\theta, f \cos\theta \rangle$

$$\vec{r}_u \times \vec{r}_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & f_u \cos\theta & f_u \sin\theta \\ 0 & -f\sin\theta & f\cos\theta \end{vmatrix}$$

$$= \Big\langle f \cdot f_u \cos^2\theta + f \cdot f_u \sin^2\theta, \; -f\cos\theta, \; -f\sin\theta \Big\rangle$$

$$= \Big\langle f \cdot f_u, -f\cos\theta, f\sin\theta \Big\rangle$$

$$= f \langle f_u, -\cos\theta, \sin\theta \rangle$$

AND $|\vec{r}_u \times \vec{r}_\theta| = f \sqrt{\left(\frac{\partial f}{\partial u}\right)^2 + \cos^2\theta + \sin^2\theta}$

$$= f(x) \sqrt{1 + \left(\frac{df}{dx}\right)^2}$$

so $SA = \iint_D f(x)\sqrt{1+\left(\frac{df}{dx}\right)^2}\, dA$

$$= \int_a^b \int_0^{2\pi} f(x)\sqrt{1+[f'(x)]^2}\, d\theta\, dx$$

$$= 2\pi \int_a^b f(x)\sqrt{1+[f'(x)]^2}\, dx$$