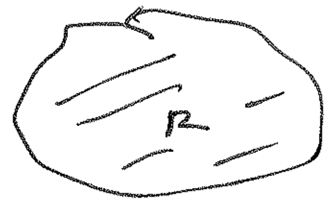
16.4: Green's Thm (MIT)

$\text{curl}(\vec{F}) = Q_x - P_y$ where $\vec{F} = \langle P, Q \rangle$

is a measure of how far a field is from conservative.

What if $\text{curl}(\vec{F}) \neq \vec{0}$ & I want

$$\oint_C \vec{F} \cdot d\vec{r} = ?$$



C

opt 1: direct calculation

opt 2: Green's Thm.

Green's Thm (avoids line Integral)

If C is a closed curve enclosing a region R, CCW, $\vec{F}$ vector field defined & diff in R then

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_R \text{curl}\,\vec{F} \cdot dA$$

or
$$\oint_C P\,dx + Q\,dy = \iint_R (Q_x - P_y)\,dA$$

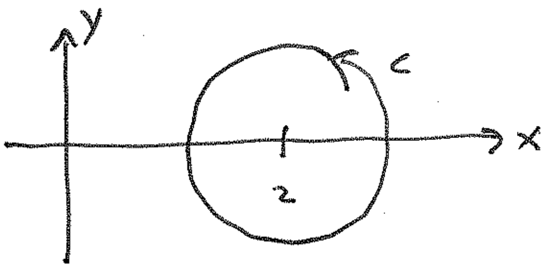
CCW due to CURL

only on the bounds

in the region.

warning: only for closed curves.

Ex1: Let C = circle w/ $r=1$ centered @ $(2,0)$



compute

$$I = \oint_C \underbrace{y e^{-x}}_{P}\, dx + \underbrace{\left(\tfrac{1}{2}x^2 - e^{-x}\right)}_{Q} dy$$

(1) directly....

$x = 2 + \cos\theta$

$y = \sin\theta$

$dx = -\sin\theta \, d\theta$

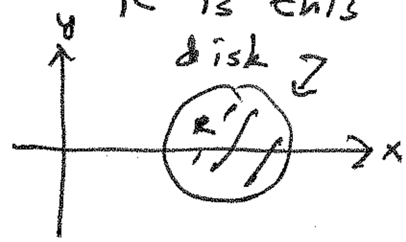
$dy = \cos\theta \, d\theta$ ... yuck.

(2) Green's Thm:

$$I = \iint_R \operatorname{curl}\vec{F}\, dA$$

$$= \iint_R (Q_x - P_y)\, dA \quad \text{where}$$

R is this disk



$$= \iint_R (\underbrace{x + e^{-x}}_{Q_x} - \underbrace{e^{-x}}_{P_y})\, dA$$

$$= \iint_R x\, dA \quad =: M_y \text{ and } \bar{x} = \frac{M_y}{m}$$

$$= \text{Area}(R)\cdot \bar{x}$$

$$= 2\pi$$

special case:

If $\text{curl}\,\vec{F} = \vec{0}$

then $\vec{F}$ conservative?

Green's Thm: $\oint_C \vec{F}\cdot d\vec{r} = \iint_R \text{curl}\,\vec{F}\, dA$

$= 0$ if $\text{curl}\,\vec{F} = \vec{0}$

This proves: if $\text{curl}\,\vec{F} = 0$ everywhere in R,
then $\oint_C \vec{F}\cdot d\vec{r} = 0$.

Consequence of Green's Thm: If $\vec{F}$ is defined everywhere in the plane & $\text{curl}\,\vec{F} = \vec{0}$, then $\vec{F}$ is conservative.

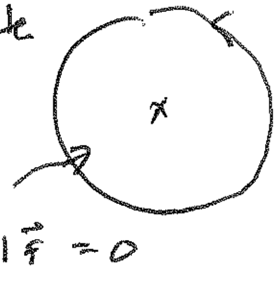
☐ proof:

$$\oint_C \vec{F} d\vec{r} = \iint_R \text{curl}\,\vec{F}\, dA$$

$$= \iint_R \vec{0}\, dA$$

$$= 0$$

unit circle



$\text{curl}\,\vec{F} = 0$
everywhere
except $(0,0)$ where
it is undefined.

Can't use Green's Thm
on problems of
this type.

Proof of Green's thm: $\oint_C P\,dx + Q\,dy = \iint_R (Q_x - P_y)\,dA$

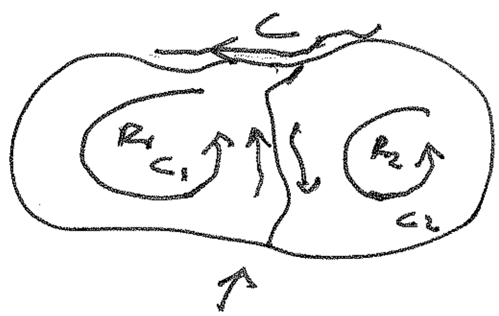
observe 1: we'll prove $\oint_C P\,dx = \iint_R -P_y\,dA$

(special case where $Q=0$.)

A similar argument will show $\oint_C Q\,dy = \iint_R Q_x\,dA$.

The sum of the formulas will prove Green's Thm!

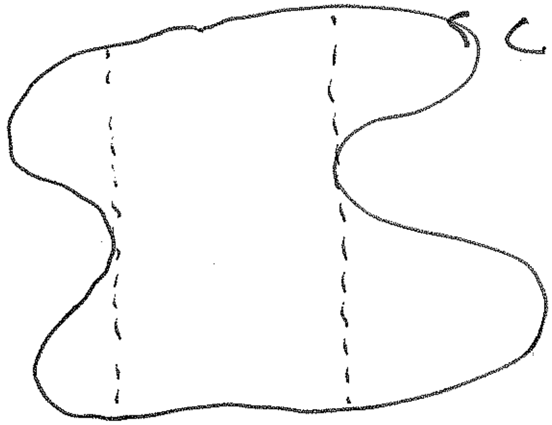
observe 2: we can decompose R into simpler regions.



If we can prove the statement is true for C_1 & C_2, then it is true for C.

notice that the middle edge is traversed twice but in opposite directions.

$$\oint_C M\,dx = \oint_{C_1} + \oint_{C_2} = \iint_{R_1} + \iint_{R_2} = \iint_R -M_y\,dA$$

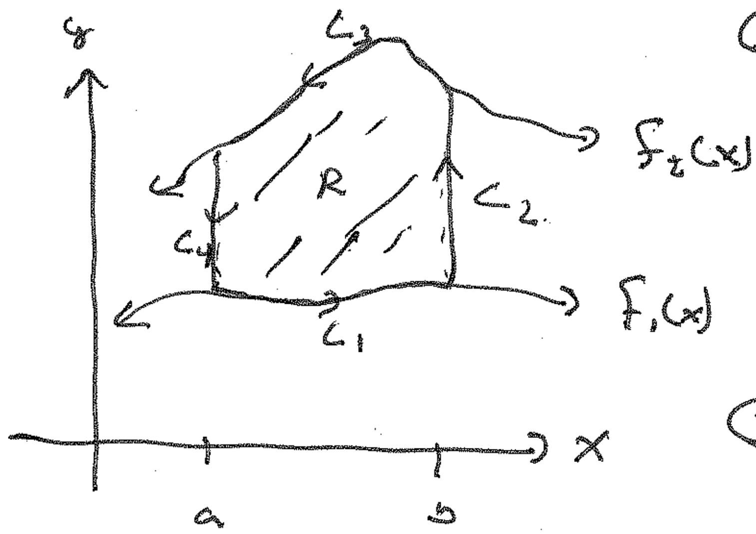


Cut R into "vertically simple regions."

$$\left\{\begin{array}{c} a < x < b \\ \text{and} \\ f_1(x) < y < f_2(x) \end{array}\right.$$

<u>Main step</u>: prove $\oint_C P\,dx = \iint_R -P_y\,dA$ if (LHS, RHS) R is vertically simple & C = boundary of R c.c.w.

$\boxed{\text{L.H.S. in 4 pieces}}$



① $\int_{C_1} P\,dx = \int_a^b P(x, f_1(x))\,dx$

$y = f_1(x)$

x from a to b.

② $\int_{C_2} P\,dx = 0$

$x = b \Rightarrow dx = 0$

③ And $\int_{C_3} P\,dx = \int_b^a P(x, f_2(x))\,dx$

$y = f_2(x)$

x from b to a

$= -\int_a^b P(x, f_2(x))\,dx$

④ and $\int_{C_4} P\,dx = 0.$

so $\oint_C P\,dx = \int_a^b P\,(x, f_1(x))\,dx - \int_a^b P\,(x, f_2(x))\,dx$

This is the L.H.S.

$\boxed{\text{Now for the R.H.S.}}$

$$\iint_R -P_y\,dA = -\int_a^b \int_{y=f_1(x)}^{y=f_2(x)} \frac{\partial P}{\partial y}\,dy\,dx$$

$$= -\int_a^b \Big[P(x, y) \Big]_{y=f_1(x)}^{y=f_2(x)} dx$$

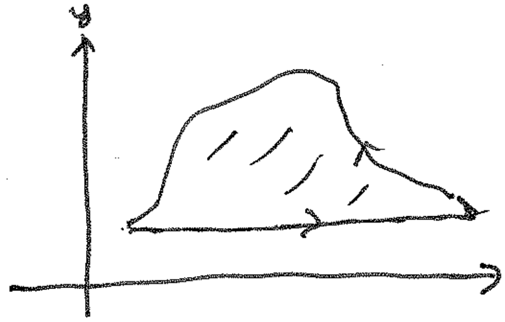
$$= -\int_a^b P(x, f_2(x)) - P(x, f_1(x))\,dx$$

$$= \int_a^b \left[P(x, f_1(x)) - P(x, f_2(x)) \right] dx$$

this is the RHS.

As you can see, LHS = RHS. We can now remove the assumption of "vertically simple" because of observation 2 & the assumption that $Q = 0$ in observation 1.

ex2: planimeter.



Area under the curve

$$\oint_C x\,dy = \iint_R 1\,dA = \text{Area}(R).$$