15.9: Change of Variables.

This section is a multivariate approach to integration by substitution.

ex: $\int_a^b \sin 2x \, dx = \int_{2a}^{2b} \sin(u) \underbrace{\tfrac{1}{2}}_{\text{scaling factor}} du$

Let's start a few examples to better see the need and sticky point

ex1: $I = \iint_R \sin(9x^2+4y^2)\, dA$ where R is the region in Q1 bounded by $9x^2+4y^2=1$.

Let $u = 3x$ and $v = 2y$

$$\Rightarrow I = \iint_{\text{unit disk}} \sin(u^2+v^2) \begin{pmatrix}\text{scaling}\\ \text{factor}\end{pmatrix} dA'$$

in terms of u & v.

ex2: $I = \iint_R \dfrac{x-2y}{3x-y}\, dA$ where R is the parallelogram enclosed by $x-2y=0$; $x-2y=4$; $3x-y=1$; $3x-y=8$

Let $u = x-2y$ and $v = 3x-y$

$$I = \int_1^8 \int_0^4 \frac{u}{v} \begin{pmatrix}\text{scaling}\\ \text{factor}\end{pmatrix} du\, dv$$

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ex3: $I = \iint_R (x+y)e^{x^2-y^2}\,dA$ where R is the rectangle enclosed by $x-y=0$; $x-y=2$

$x+y=0$; $\cancel{x}+y=3$

Let $u = x-y$ and $v = x+y$

$$\Rightarrow I = \int_0^3 \int_0^2 v e^{uv} \begin{pmatrix}\text{scaling}\\ \text{factor}\end{pmatrix} du\,dv$$

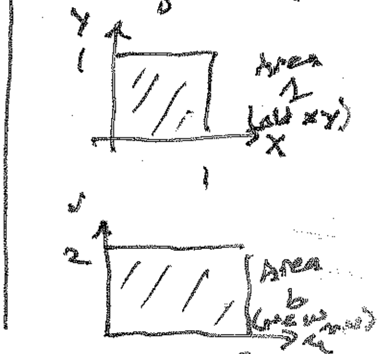
ex1 rev: $I = \iint_R \sin(9x^2+4y^2)\,dA$ where R is the region in Q1 bounded by $9x^2+4y^2=1$.

Let $u = 3x$ and $v = 2y$

$\Rightarrow \quad x = \frac{u}{3}$ and $y = \frac{v}{2}$

$$\Rightarrow \quad \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1/3 & 0 \\ 0 & 1/2 \end{vmatrix} = \frac{1}{6}$$

Find geometric scaling factor



$$\Rightarrow I = \frac{1}{6}\iint_D \sin(u^2+v^2)\,dA'$$

where A' is the unit disk in Q1 for (u,v).

$$= \frac{1}{6}\int_0^{\frac{\pi}{2}}\int_0^1 \sin(r^2)\,r\,dr\,d\theta$$

$$= \frac{1}{6}\int_0^{\frac{\pi}{2}} \left[-\frac{1}{2}\cos(r^2)\right]_0^1 d\theta$$

$$= -\frac{1}{12}\int_0^{\pi/2} (\cos(1) - 1)\,d\theta$$

$$= \frac{\pi}{24}(1-\cos(1))$$

Scaling factor $= \frac{\text{old}}{\text{new}}$

$= \frac{1}{6}$

$\boxed{\begin{matrix}15.9 \\ 3/11\end{matrix}}$

$I = \iint_R \frac{x-2y}{3x-y}\, dA$ where R is the parallelogram enclosed by $x - 2y = 0$; $x - 2y = 4$

$3x - y = 1$; $3x - y = 8$

Let $u = x - 2y$ and $v = 3x - y$

$$I = \int_1^8 \int_0^4 \frac{u}{v} \left(\underset{\text{factor}}{\text{scaling}}\right) du\, dv$$

$$= \frac{1}{5}\int_1^8 \left[\frac{1}{2}\frac{u^2}{v}\right]_0^4 dv$$

$$= \frac{1}{10}\int_1^8 \frac{16}{v}\, dv$$

$$= \frac{8}{5}\left[\ln|v|\right]_1^8$$

$$= \frac{8}{5}\ln 8.$$

The scaling factor.

(I) $\underbrace{\begin{bmatrix} 1 & -2 \\ 3 & -1 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix}$

$\det(A) = 5$

so the (u,v) rectangle is 5 times larger than the (x,y) parallelogram.

(II) $x = -\frac{1}{5}u + \frac{2}{5}v$

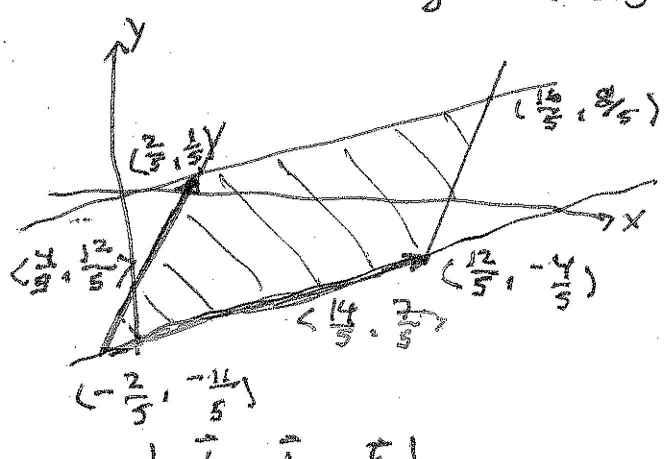
$y = -\frac{3}{5}u + \frac{1}{5}v$

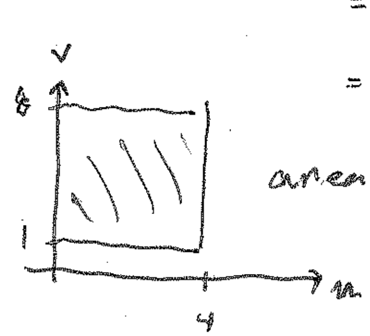
$$\Rightarrow \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} -\frac{1}{5} & \frac{2}{5} \\ -\frac{3}{5} & \frac{1}{5} \end{vmatrix}$$

$$= -\frac{1}{25} + \frac{6}{25}$$

$$= \frac{1}{5}$$

Geometric Scaling Factors





area $= 28$ [new uv]

area: $\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 14/5 & 7/5 & 0 \\ 4/5 & 12/5 & 0 \end{vmatrix} = \langle 0, 0, \frac{140}{25} \rangle$

area $= \frac{28}{5}$ (old xy)

$$\begin{matrix}\text{Scaling} \\ \text{Factor}\end{matrix} = \frac{\text{old}}{\text{new}} = \frac{28/5}{28} = \frac{1}{5}$$

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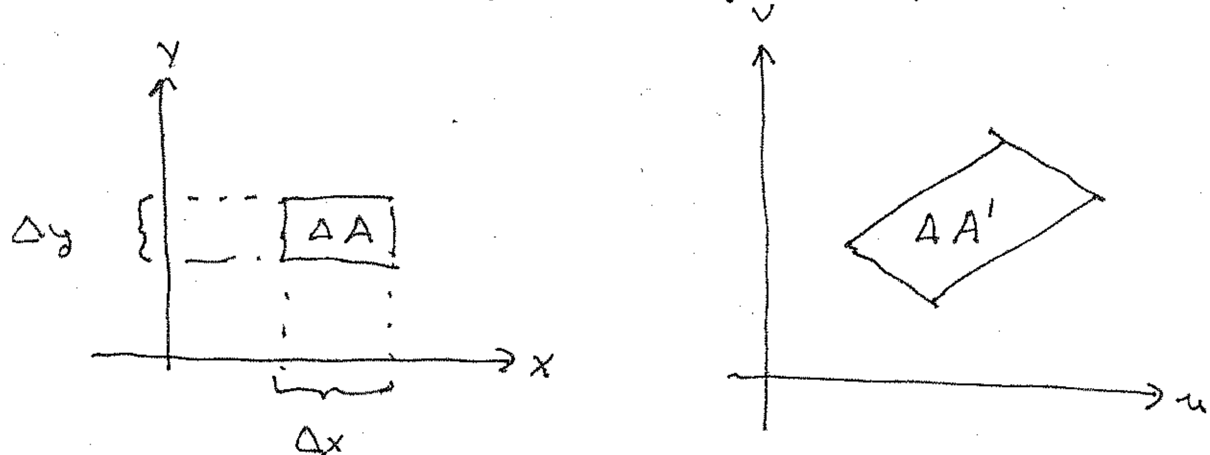
Scaling Factor Derivation (Linear)

For a substitution $u = u(x,y)$ and $v = v(x,y)$, what is the scaling factor $dx\,dy$ vs. $du\,dv$?

Suppose $u = 3x - 2y$, $v = x + y$. (to simplify the integrand or the bounds).

Find the relation between $dA = dx\,dy$ & $dA' = du\,dv$. (exchange rate).



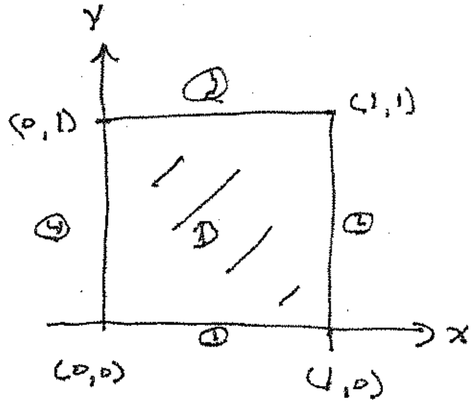
a parallelogram since linear trans *

We should have a constant scaling factor that is independent of the choice of rect.

So let's use the unit square.

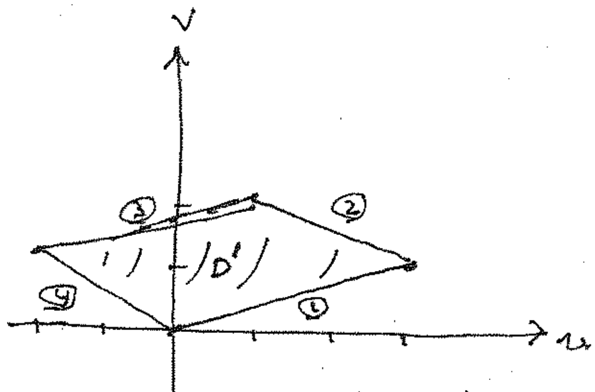
$$* \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 3x - 2y \\ x + y \end{bmatrix} = x\begin{bmatrix} 3 \\ 1 \end{bmatrix} + y\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = A\vec{x}$$

so it's a linear transform-ation.



$A = 1$

For any rectangle, ΔA is $\frac{1}{5}\Delta A' \Rightarrow dA' = 5\,dA$
$du\,dv = 5\,dx\,dy$



$A' = \begin{vmatrix} 3 & 1 \\ -2 & 1 \end{vmatrix} = 5$

Q: why does the determinant give the area?

$$\hookrightarrow \quad \iint_D f\,dx\,dy = \iint_{D'} f'\,\frac{1}{5}\,du\,dv$$

in terms of x & y | in terms of u & v

it isn't always a const. scaling factor, because the transformation $T: (x,y) \to (u,v)$ isn't necessarily linear. But we can address this thru linear approximation.

$I = \iint_R (x+y)e^{x^2-y^2}\,dA$ where R is the rectangle enclosed by $x-y=0$; $x-y=2$
$x+y=0$; $x+y=3$

Let $u = x-y$ and $v = x+y$

$$\Rightarrow \quad I = \int_0^3\int_0^2 v e^{uv}\begin{pmatrix}\text{scaling}\\ \text{factor}\end{pmatrix} du\,dv$$

$$\begin{tikzcd}
(x,y) \arrow[rr, bend left, "\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}"] & & (u,v) \arrow[ll, bend left, "\frac{1}{2}\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}"]
\end{tikzcd}$$

Scaling factor of $\frac{1}{2}$

$$I = \int_0^3\int_0^2 v e^{uv}\tfrac{1}{2}\,du\,dv$$

$$= \frac{1}{2}\int_0^3\left[e^{uv}\right]_0^2 dv$$

$$= \frac{1}{2}\int_0^3 (e^{2v}-1)\,dv$$

$$= \frac{1}{2}\left[\frac{1}{2}e^{2v}-v\right]_0^3$$

$$= \frac{1}{2}\left(\frac{1}{2}e^6 - 3 - \frac{1}{2} + 0\right)$$

$$= \frac{1}{4}e^6 - \frac{7}{4}$$

Note: If you switch u & v ... that is $u = x+y$
$v = x-y$

we find the Jacobian

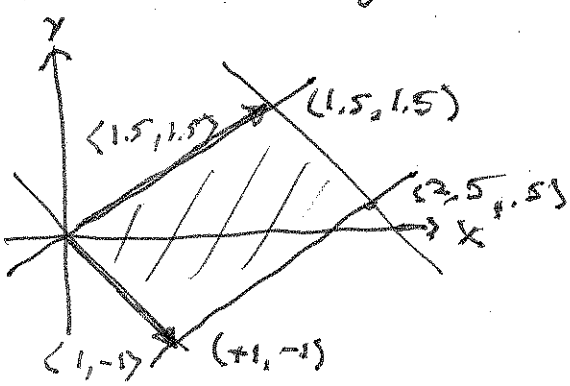
$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix}1/2 & 1/2\\ 1/2 & -1/2\end{vmatrix} = -\frac{1}{2}.$$

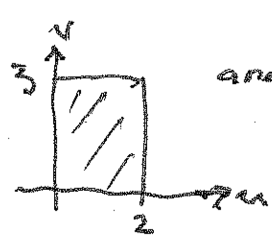
but the scaling factor is the same $\left|-\frac{1}{2}\right| = \frac{1}{2}$.

Geometric Scaling Factor

Area: $\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 1.5 & 1.5 & 0\\ 1 & -1 & 0\end{vmatrix} = \langle 0, 0, -3\rangle$

area $= 3$ (old xy)





area $= 6$ (new uv)

$$\begin{matrix}\text{scaling}\\ \text{factor}\end{matrix} = \frac{\text{old}}{\text{new}} = \frac{3}{6} = \frac{1}{2}$$

General case: (differentials).

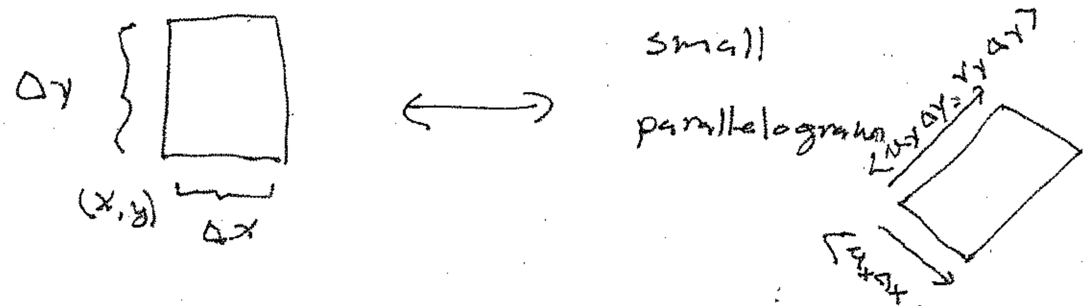
$\boxed{\begin{matrix}15.9\\ 21/11\end{matrix}}$

$u = u(x,y) \qquad \Delta u \approx u_x \Delta x + u_y \Delta y$

$v = v(x,y) \qquad \Delta v \approx v_x \Delta x + v_y \Delta y$

$$\text{or} \quad \begin{bmatrix}\Delta u \\ \Delta v\end{bmatrix} \approx \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} \begin{bmatrix}\Delta x \\ \Delta y\end{bmatrix}$$

So a small rect. in (x,y) has the image of a small parallelogram under T



same argument ... the determinant tells us how we scaled coords.

$\langle \Delta x, 0\rangle \rightsquigarrow \langle \Delta u, \Delta v\rangle \approx \langle u_x \Delta x, v_x \Delta x\rangle$

sides of par.

$\langle 0, \Delta y\rangle \rightsquigarrow \langle \Delta u, \Delta v\rangle \approx \langle u_y \Delta y, v_y \Delta y\rangle$

the area = det( ) dx dy.

<u>Jacobian</u>: $J = \dfrac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$

Then $du\,dv = \left|\dfrac{\partial(u,v)}{\partial(x,y)}\right| dx\,dy$

Find the scaling factor

$x = r\cos\theta \quad ; \quad y = r\sin\theta \quad ; \quad z = z$

$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= 1 \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix}$$

$$= r$$

so $\underbrace{dx\,dy\,dz}_{\text{cartesian}} = \underbrace{r\,dr\,d\theta\,dz}_{\text{spherical.}}$ (in no particular order)

Find the scaling factor

$x = \rho \sin(\phi)\cos(\theta)$, $y = \rho \sin(\phi)\sin(\theta)$, $z = \rho\cos(\phi)$

$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}$$

$$= \begin{vmatrix} \sin(\phi)\cos(\theta) & -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \sin(\phi)\sin(\theta) & \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \\ \cos(\phi) & 0 & -\rho\sin\phi \end{vmatrix}$$

$$= \cos\phi \begin{vmatrix} -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \end{vmatrix}$$

$$-\rho\sin\phi \begin{vmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta \end{vmatrix}$$

$$= -\rho^2\cos\phi\left(\sin\phi\cos\phi(\sin^2\theta + \cos^2\theta)\right)$$

$$-\rho^2\sin^3\phi(\cos^2\theta + \sin^2\theta)$$

$$= -\rho^2\sin\phi(\cos^2\phi + \sin^2\phi)$$

$$= -\rho^2\sin\phi$$

The scaling factor is the magnitude of the Jacobian.

$dV = \rho^2\sin\phi \, d\rho \, d\theta \, d\phi$

HARD EXAMPLE

$I = \iint_R x^2 - xy + y^2 \, dA$ where R is the region bounded by the ellipse $x^2 - xy + y^2 = 2$ $\boxed{\begin{matrix} 15.9 \\ 10/11 \end{matrix}}$

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In[13]:= Show[{Plot[{x, -x}, {x, -3, 3}],
   ContourPlot[x^2 - x y + y^2 == 2, {x, -3, 3}, {y, -3, 3}, Axes → True]}, AspectRatio → 1]
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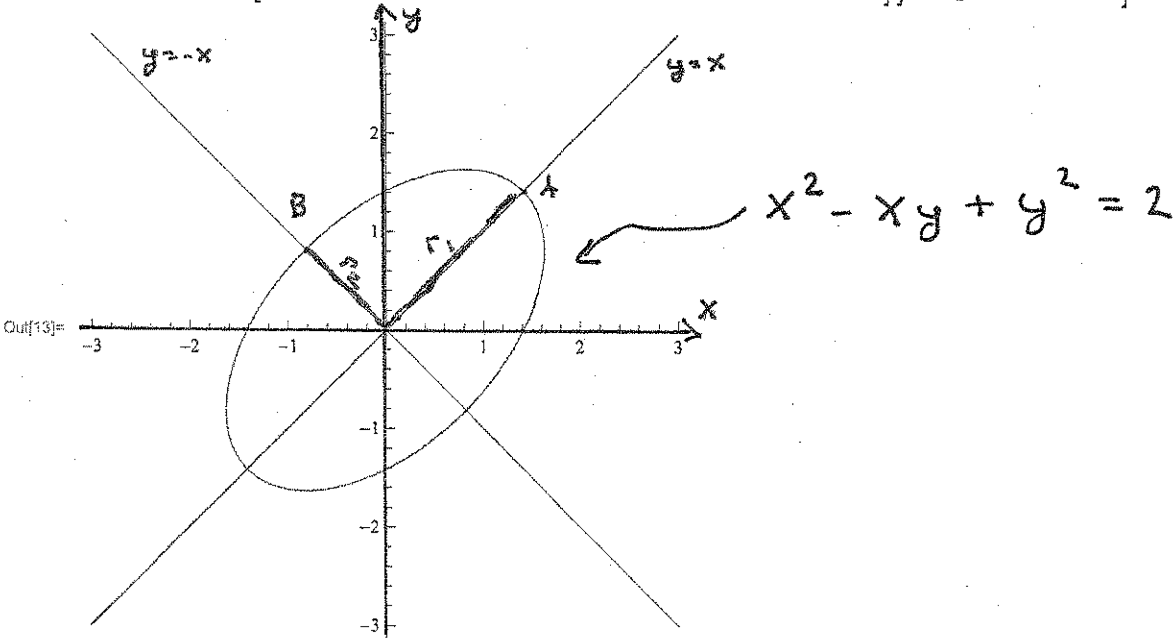
Out[13]=



It's an ellipse rotated by $\frac{\pi}{4}$ c.c.w and scaled.

How is it scaled?

Find (A). $x^2 - x^2 + x^2 = 2$ (since $y = x$)

$\Rightarrow x = \pm\sqrt{2}$

$\Rightarrow A(\sqrt{2}, \sqrt{2})$ $\longrightarrow$ major axis has length $2 \cdot 2$

$r_1 = 2$

Find (B). $x^2 + x^2 + x^2 = 2$

$\Rightarrow x = \pm\sqrt{\frac{2}{3}}$

$\Rightarrow B\left(-\sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}\right)$ $\longrightarrow$ minor axis has length $2 \cdot \frac{2}{\sqrt{3}}$

$r_2 = \frac{2}{\sqrt{3}}$

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our new variables (u, v) should be s.t.

$$x^2 - xy + y^2 = 2 \longrightarrow u^2 + v^2 = 1$$

$$(u,v) = \begin{array}{c}\text{compress} \\ x \text{ by } \frac{1}{2} \\ \text{stretch} \\ y \text{ by } \sqrt{3}/2\end{array} \leftarrow \begin{array}{c}\text{rotate} \\ \pi/4 \\ \text{c.w}\end{array} \leftarrow (x,y)$$

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 1/2 & 0 \\ 0 & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} \cos(-\frac{\pi}{4}) & -\sin(-\frac{\pi}{4}) \\ \sin(\frac{\pi}{4}) & \cos(\frac{\pi}{4}) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\Rightarrow u = \frac{1}{2\sqrt{2}}(x+y) \quad \text{and} \quad v = \frac{\sqrt{3}}{2\sqrt{2}}(y-x)$$

and $x = \sqrt{2}\,u - \sqrt{\frac{2}{3}}\,v$ and $y = \sqrt{2}\,u + \sqrt{\frac{2}{3}}\,v$

Integrand: $x^2 - xy + y^2 = 2(u^2 + v^2)$

Region: $x^2 - xy + y^2 \leq 2 \Rightarrow u^2 + v^2 \leq 1$

Scaling factor: $\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \sqrt{2} & \sqrt{2} \\ -\sqrt{\frac{2}{3}} & \sqrt{\frac{2}{3}} \end{vmatrix} = \frac{4}{\sqrt{3}}$

$$\Rightarrow I = \iint_{u^2+v^2 \leq 1} 2(u^2+v^2)\,\frac{4}{\sqrt{3}}\,du\,dv$$

$$= \int_0^{2\pi}\int_0^1 2r^2 \cdot \frac{4}{\sqrt{3}}\,r\,dr\,d\theta$$

$$= 2\pi \cdot \frac{4}{\sqrt{3}} \cdot 2 \cdot \frac{1}{4}$$

$$= \frac{4\pi}{\sqrt{3}}$$